On the Hyperbolicity Criterion for Noncompact Riemannian Manifolds of Special Type

A. G. Losev

ABSTRACT. In this paper we study the behavior of bounded harmonic functions on complete Riemannian manifolds (of a certain special type) depending on the geometry of the manifold.

Here we study how the behavior of bounded harmonic functions on complete Riemannian manifolds of a certain special type depends on the geometry of the manifold.

The classical Liouville theorem states that any bounded harmonic function on \( \mathbb{R}^n \) is a constant. But there is an extensive class of manifolds on which nontrivial bounded harmonic functions exist. For example, M. Anderson [1] and D. Sullivan [2] solved the Dirichlet problem for complete connected manifolds with sectional curvature bounded by two negative constants. Developing this question further, S. T. Yau [3] presented the following classification of Riemannian manifolds.

**Definitions.** Let us call a manifold **strongly parabolic** if any positive harmonic function on it is a constant. A manifold is called **parabolic** if the set of positive harmonic functions on it is of finite dimension. All other manifolds are called **hyperbolic,** i.e., nonparabolic.

In this paper we give a hyperbolicity criterion for complete Riemannian manifolds of a certain special type.

Let \( M \) be a complete Riemannian manifold that can be represented as the union \( M = B \cup D \), where \( B \) is a compact set and \( D \) is isometric to the direct product \( \mathbb{R}_+ \times S \) (where \( \mathbb{R}_+ = (0, \infty) \) and \( S \) is a compact Riemannian manifold) with the metric

\[
ds^2 = h^2(r) \, dr^2 + g^2(r) \, d\theta^2.
\]

Here \( h(r) \) and \( g(r) \) are smooth positive functions on \( \mathbb{R}_+ \), and \( d\theta^2 \) is a metric on \( S \). Examples of such spaces are Euclidean space \( (h(r) = 1, \ g(r) = r) \), Lobachevski space \( (h(r) = 1, \ g(r) = sh(r)) \), a surface obtained by the rotation of a graph of a function \( f(r) \) around the ray \( Or \) in \( \mathbb{R}^n \) \( (h(r) = \sqrt{1 + [f'(r)]^2}, \ g(r) = f(r)) \), and others.

Let us introduce the notation

\[
I = \int_{r_0}^{\infty} h(t)g^{1-n}(t) \left( \int_{r_0}^{t} h(\xi)g^{n-3}(\xi) \, d\xi \right) \, dt,
\]

where \( r_0 = \text{const} > 0 \), \( n = \text{dim} \, M \).

**Theorem.** 1) Suppose that \( I = \infty \) for a Riemannian manifold \( M \). Then \( M \) is strongly parabolic.

2) Suppose that \( I < \infty \) for a Riemannian manifold \( M \). Then \( M \) is hyperbolic. Moreover, for any function \( \Phi(\theta) \) continuous on \( S \) there is a harmonic function \( u \) such that

\[
\lim_{r \to \infty} u(r, \theta) = \Phi(\theta).
\]
Remark 1. The condition $I < \infty$ implies that

$$\int_{r_0}^{\infty} h(t) g^{1-n}(t) \, dt < \infty.$$ 

The last inequality is equivalent to the existence of Green's function on the manifold $M$ (cf. [4]).

**Proof of the Theorem.** The first statement of the Theorem is proved in the paper [4]. Let us prove the second statement. Note that in the coordinates $(r, \theta)$ the Laplace-Beltrami operator has the form

$$\Delta = \frac{1}{h^2(r)} \frac{\partial^2}{\partial r^2} + \frac{1}{h^2(r)} \left( (n-1) \frac{g'(r)}{g(r)} - \frac{h'(r)}{h(r)} \right) \frac{\partial}{\partial r} + \frac{1}{g^2(r)} \Delta_{\theta},$$

where $\Delta_{\theta}$ is the inner Laplacian on $S$. This formula is checked directly by the definition of the Laplace-Beltrami operator (also cf. [4]).

Let $w_k$ be an orthonormal basis of $L^2(S)$ consisting of eigenfunctions of the Laplace operator $\Delta_{\theta}$, and $\lambda_k$ be the corresponding eigenvalues. Then for any $r$ we have

$$u(r, \theta) = \sum_{k=0}^{\infty} v_k(r) w_k(\theta),$$

where

$$v_k(r) = \int_S u(r, \theta) w_k(\theta) \, d\theta, \quad \Delta_{\theta} w_k(\theta) + \lambda_k w_k(\theta) = 0.$$ 

Formula (1) implies that for any $k$ the function $v_k(r)$ is a solution of the following ordinary differential equation

$$v''(r) + \left( (n-1) \frac{g'(r)}{g(r)} - \frac{h'(r)}{h(r)} \right) v'(r) - \lambda \frac{h^2(r)}{g^2(r)} v(r) = 0,$$

where $\lambda = \lambda_k$. Equation (2) is equivalent to the equation

$$\left( \frac{g^{n-1}(r)}{h(r)} v'(r) \right)' = \lambda g^{n-3}(r) h(r) v(r).$$

Let us integrate this equation from $r_0$ to $r$. We obtain

$$v'(r) = \lambda \frac{h(r)}{g^{n-1}(r)} \int_{r_0}^{r} g^{n-3}(\xi) h(\xi) v(\xi) \, d\xi + \frac{v'(r_0) g^{n-1}(r_0)}{h(r_0)} \frac{h(r)}{g^{n-1}(r)}.$$  

Integrating once more from $r_0$ to $r$, we get

$$v(r) = \lambda \int_{r_0}^{r} \frac{h(t)}{g^{n-1}(t)} \left( \int_{r_0}^{t} g^{n-3}(\xi) h(\xi) v(\xi) \, d\xi \right) \, dt + \frac{v'(r_0) g^{n-1}(r_0)}{h(r_0)} \int_{r_0}^{r} \frac{h(t)}{g^{n-1}(t)} \, dt + v(r_0).$$

Let us denote by $l_k(r)$ the solution of (2) with $\lambda = \lambda_k$ satisfying the boundary conditions $v(r_0) = 1$, $v'(r_0) = 0$. Then (3) evidently implies that $l_k(r)$ is a monotone increasing positive function.

Let us show now that $l_k(r)$ is bounded on $(r_0, +\infty)$. Since $l_k(r)$ is positive and monotone increasing on $(r_0, +\infty)$, (3) implies that

$$l_k'(r) \leq \lambda_k g^{1-n}(r) h(r) l_k(r) \int_{r_0}^{r} h(\xi) g^{n-3}(\xi) \, d\xi.$$ 

Hence

$$(\ln l_k(r))' \leq \lambda_k \frac{h(r)}{g^{n-1}(r)} \int_{r_0}^{r} h(\xi) g^{n-3}(\xi) \, d\xi.$$