Finite-Dimensional Discrete Linear Stochastic Accelerated-Time Systems and Their Application to Quadratic Stochastic Dynamical Systems

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ABSTRACT. The limiting behavior of the trajectories \( \{x^{(n)}\} \) of linear discrete stochastic systems of the form 
\[ (K, P^{a^n+b})_{n \in \mathbb{N}}, \]
where \( K \) is the standard simplex in \( \mathbb{R}^N \), \( P : \mathbb{R}^N \to \mathbb{R}^N \) is a linear operator, \( PK \subseteq K \), 
\( a \in \mathbb{N}, \ b \in \mathbb{Z}, \ a + b > 0 \), is described. An application to a class of quadratic stochastic dynamical systems is considered.

In the theory of homogeneous Markov chains with a finite number of states \([1, 2]\), in mathematical models of migration \([3, p. 576 of the Russian translation]\), and in other areas of mathematics, the following type of finite-dimensional discrete dynamical systems naturally emerge. Let \( P : \mathbb{R}^N \to \mathbb{R}^N \) be the linear operator defined with respect to the standard frame \( (e_j)_{j=1}^N \) in the space \( \mathbb{R}^N \) by the matrix \( (p_{ij})_{i,j=1}^N \) whose elements satisfy the relations
\[
p_{ij} \geq 0, \quad \sum_{i=1}^N p_{ij} = 1. \tag{1}
\]
An operator \( P \) of this form will be called a stochastic linear operator. Let \( K \) be the standard simplex in \( \mathbb{R}^N \):
\[
K = \left\{ x = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}_+^N : \sum_{j=1}^N \xi_j = 1 \right\}.
\]
Obviously, conditions (1) ensure that \( K \) is invariant under \( P \), and hence the action of the semigroup of the powers of \( P \) on the simplex \( K \) endowed with the \( l_1^N \)-metric is defined. Thus, we have defined the topological semigroup discrete-time dynamical system
\[
(K, P^n)_{n \in \mathbb{N}^+}. \tag{2}
\]
As usual, the trajectory \( \{x^{(n)}\}_{n \in \mathbb{N}} \) of a point \( x \in K \) is determined by the relation \( x^{(n)} = P^nx \); the set of its limit points will be denoted by \( \omega(x) \). One of the main questions studied in the theory of topological dynamical systems is the limiting behavior of their trajectories. The description of this behavior for the situation outlined above is known; we present it below in convenient form.

The goal of this work is to examine the limiting behavior of the trajectories of the following temporally inhomogeneous system generated by the homogeneous dynamical system (2):
\[
(K, P^{a^n+b})_{n \in \mathbb{N}}, \tag{3}
\]
where \( a \in \mathbb{N}, \ a \neq 1, \ b \in \mathbb{Z}, \ a + b > 0 \). This system will be called a finite-dimensional discrete accelerated-time linear stochastic system.

As an application, we shall consider the case of quadratic stochastic operators, some of which lead to accelerated-time systems.

But first let us review the classical case. It is well known \([4, \text{Chap. XIII}]\) that for any dynamical system (2) there exists a number \( m \in \mathbb{N} \) such that any subsequence
\[
\{x^{(mq+r)}\}_{q=0}^{\infty}, \quad (r = 0, 1, \ldots, m-1)
\]
converges and, in particular, \( \omega(x) \) is a finite set for any \( x \in K \). There are many modern proofs of this result, supplied with formulations convenient for our purposes. Here is one of these formulations [5, p. 115]. Let \( P \) be a stochastic linear operator in \( \mathbb{R}^N \) supplied with the \( l^1_0 \)-norm. Then \( \mathbb{R}^N \) can be decomposed into the direct sum of subspaces

\[
\mathbb{R}^N = E_0 \oplus E_1
\]
such that

1) \( PE_0 \subseteq E_0 \) and \( PE_1 \subseteq E_1 \);
2) \( r(P_0) < 1 \), where \( r(P_0) \) is the spectral radius of the operator \( P \) restricted to the invariant subspace \( E_0 \); in particular, \( \lim_{n \to \infty} P_0^n = 0 \);
3) for a certain positive integer \( m \), the operators \( P_1, P_1^2, \ldots, P_1^m \); where \( P_1 \) is the restriction of \( P \) to \( E_1 \), are pairwise distinct isometries of \( E_1 \), while \( P_1^m \) is the identity operator in \( E_1 \), so that the equation

\[
P_1^{mq+r} = P_r
\]
holds for all \( q, r \in \mathbb{N} \);
4) \( \lim_{q \to \infty} P_{mq+r} = P_1^r \).

The smallest number \( m \) satisfying condition 3 will be denoted by \( \text{ind} \, P \).

The following lemma (together with the description of the limiting behavior of trajectories given above) provides the basis for describing similar objects in accelerated-time systems. It is formulated in terms of congruences in \( \mathbb{Z} \). Let us recall this notion and the relevant notation [6, Chap. 3] (see also [7]).

The integers \( a \) and \( b \) are said to be congruent modulo \( m \), where \( m \in \mathbb{Z}, m \neq 0 \), if \( b-a \) is divisible by \( m \). This is written \( a \equiv b(m) \). A congruence modulo \( m \) is an equivalence relation in \( \mathbb{Z} \). For any \( a \in \mathbb{Z} \), the set of the integers congruent to \( a \) modulo \( m \) is denoted by \( \overline{a} \) and is called a residue class, its elements are called residues modulo \( m \). The greatest common divisor of integers \( a \) and \( b \) is denoted by \( (a,b) \); in particular, \( a \) is coprime to \( b \) if \( (a,b) = 1 \). An integer \( a \) is said to be of order \( \gamma \) modulo \( m \), where \( (a,m) = 1 \), if \( \gamma \) is the least positive integer such that \( a^\gamma \equiv 1(m) \). Let us mention some useful properties of the order.

1. If \( a \) is of order \( \gamma \) modulo \( m \), then no two of the numbers \( 1 = a^0, a^1, \ldots, a^{\gamma-1} \) are congruent modulo \( m \).
2. The congruence \( a^\delta \equiv 1(m) \) is valid if and only if \( \delta \) is divisible by \( \gamma \).

**Lemma.** For any two positive integers \( a, m \),

1) there is a unique pair of numbers \( t, k \in \mathbb{N} \) such that

\[
a^{k+t} \equiv a^k(m),
\]
and \( k+t \) is the least number \( n \in \mathbb{N} \) that satisfies the congruence \( a^n \equiv a^r(m) \) for a certain positive integer \( r < n \);
2) for any positive integer \( n \geq k \) and any \( d \in \mathbb{Z}_+ \), we have

\[
a^{n+dt} \equiv a^n(m);
\]
3) if \( n \geq t \) and \( i \) is the remainder of \( n \) on division by \( t \), then

\[
a^{k+n} \equiv a^{k+i}(m);
\]
4) the first \( k+t-1 \) terms of the sequence \( \{\overline{a^n}\}_{n \in \mathbb{N}} \) are pairwise distinct (and hence \( k+t-1 \leq m \)), while all the rest are cyclic with period \( t \) in the set

\[
\{\overline{a^k}, \overline{a^{k+1}}, \ldots, \overline{a^{k+t-1}}\};
\]
5) if \( a^n \not\equiv 0(m) \) for \( n \in \mathbb{N} \), then \( t \) is equal to the order of \( a \) modulo \( m_0 \), where \( m_0 \) is equal to \( m \) divided by the greatest term \( q \) of the sequence

\[
q_1 = (m,a), q_2 = (m,a^2), \ldots, q_n = (m,a^n), \ldots,
\]