On the Number of Summands in the Asymptotic Formula for the Number of Solutions to Waring’s Equation

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ABSTRACT. In the paper, an estimate of the number of summands in the asymptotic formula for the number of solutions to Waring’s equation is obtained. This is achieved by means of a recurrent process leading to a greater reduction than that in Vinogradov’s mean value theorem.

KEY WORDS: Waring’s problem, Waring’s equation, estimate of the number of summands, Fourier coefficients, Hardy’s equation, Vinogradov’s mean value theorem.

§1. Introduction

Let \( N_k(P)(\lambda_1, \ldots, \lambda_n) \) be the number of solutions of the system

\[
\begin{align*}
  x_1 + \cdots + x_k - y_1 - \cdots - y_k &= \lambda_1, \\
  \vdots & \\
  x^n_1 + \cdots + x^n_k - y^n_1 - \cdots - y^n_k &= \lambda_n, \\
  0 & \leq x_1, \ldots, y_k < P.
\end{align*}
\]

(Here and subsequently the variables take only integer values.) By Vinogradov’s mean value theorem, for \( k \geq nr \) the following estimate holds:

\[
N_k(P) = N_k(P)(0, \ldots, 0) \ll P^{2k-n^2/3} + P^{2k-n^2} (1 - \frac{1}{2})^r. \tag{1}
\]

Denote by \( I_k(P) \) the number of solutions of Hardy’s equation

\[
x^n_1 + \cdots + x^n_k - y^n_1 - \cdots - y^n_k = 0, \quad 0 \leq x_1, \ldots, y_k < P.
\]

Formula (1) allows us to obtain the following estimate for \( I_k(P) \):

\[
I_k(P) \ll \sum_{|\lambda_1| < kP} \cdots \sum_{|\lambda_{n-1}| < kP} N_k(P)(\lambda_1, \ldots, \lambda_{n-1}, 0) \ll P^{n(n-1)/2} N_k(P) \ll P^{2k-n+n^2} (1 - \frac{1}{2})^r. \tag{2}
\]

Let \( I(N) \) be the number of solutions to Waring’s equation

\[
x^n_1 + \cdots + x^n_k = N, \quad 0 \leq x_1, \ldots, x_k \leq N^{1/n}.
\]

In [1] it was proved that for \( n \geq 4 \) and \( k \geq 2[n^2(2\ln n + \ln \ln n + 5)] \) the following asymptotic formula is valid:

\[
I(N) = \sigma(N) \gamma N^{k/n-1} + O(N^{k/n-1-1}/(20^n \ln n)), \tag{3}
\]

where \( \gamma = (\Gamma(1 + 1/n))^k/\Gamma(k/n) \), \( \sigma(N) \geq c_0(n, k) > 0 \).

In the present paper we prove that for \( k \geq n(n-1) + nr \) the following estimate holds:

\[
I_k(P) \ll P^{2k-n+n^2} (1 - \frac{1}{2})^r.
\]

This result refines the estimate (2); this helps prove that formula (3) is valid for

\[
n \geq 4 \quad \text{and} \quad k \geq 2[n^2(\ln n + \ln \ln n + 6)].
\]
§2. Properties of the Fourier coefficients of some functions

In [2] the following assertion was proved.

Lemma 1. Suppose \( N_1, \ldots, N_n \) are nonnegative integers and \( F(\alpha_1, \ldots, \alpha_n) \) is a nonnegative real function defined on the cube \( E_n = [0, 1]^n \) and Lebesgue integrable. Suppose that the Fourier coefficients \( c(\lambda_1, \ldots, \lambda_n) \) of \( F(\alpha_1, \ldots, \alpha_n) \) are also nonnegative real numbers. Then, for any integers \( \mu_1, \ldots, \mu_n \), the following inequality is valid:

\[
\sum_{|\lambda_1| \leq N_1} \cdots \sum_{|\lambda_n| \leq N_n} c(\lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n) \leq 4^n \sum_{|\lambda_1| \leq N_1} \cdots \sum_{|\lambda_n| \leq N_n} c(\lambda_1, \ldots, \lambda_n).
\]

In subsequent arguments we shall need Lemmas 2 and 3, whose proof is similar to that of Lemma 1.

Lemma 2. Suppose that \( N \) is a nonnegative real number, \( a \) and \( b \) are integers, and \( q \) is a positive integer. Suppose also that \( F(\alpha) \) is a nonnegative real function that can be expressed as a finite Fourier series with nonnegative coefficients \( c(\lambda) \). Then the following inequalities are valid:

\[
\sum_{|\lambda| \leq N} c(a\lambda + b) \leq 4q \int_0^1 F(\alpha) \Phi(\alpha) \, d\alpha \leq 4q \sum_{|\lambda| \leq Nq^{-1}} c(aq\lambda),
\]

where

\[
\Phi(\alpha) = \sum_{|\lambda| \leq Nq^{-1}} \left(1 - \frac{|\lambda|}{[Nq^{-1}] + 1}\right)e^{-2\pi i a q\lambda} \geq 0.
\]

Proof. Let us prove the lemma in the case \( a = 1 \). For arbitrary \( a \), the proof is the same. Let \( N_1 = [Nq^{-1}] + 1 \). Then

\[
\sum_{|\lambda| \leq N} c(\lambda + b) \leq \sum_{\mu=0}^{q-1} \sum_{|\lambda| \leq N} c(q\lambda + \mu + b) = \sum_{\mu=0}^{q-1} \sigma(\mu),
\]

where

\[
\sigma(\mu) = \sum_{|\lambda| \leq N} c(q\lambda + \mu + b).
\]

Let us estimate \( \sigma(\mu) \):

\[
s(\mu) = \frac{1}{N_1^2} \sum_{x,y=0}^{N_1} \sum_{|\lambda + x - y| \leq N_1} c(q\lambda + x - y + \mu + b) \leq \frac{1}{N_1^2} \sum_{x,y=0}^{N_1} \sum_{|\lambda| \leq Nq^{-1}} c(q(\lambda + x - y) + \mu + b)
\]

\[
= \frac{1}{N_1^2} \sum_{x,y=0}^{N_1} \sum_{|\lambda| \leq Nq^{-1}} \int_0^1 F(\alpha) e^{-2\pi i q(\lambda + x - y) + \mu + b)} \, d\alpha
\]

\[
= \frac{1}{N_1^2} \int_0^1 F(\alpha) \left| \sum_{x=1}^{N_1} e^{-2\pi i q\lambda x} \right|^2 \sum_{|\lambda| \leq Nq^{-1}} e^{-2\pi i q(\lambda + \mu + b)} \, d\alpha.
\]

Estimating the last sum trivially, we obtain

\[
\sigma(\mu) \leq \frac{4}{N_1} \int_0^1 F(\alpha) \left| \sum_{x=1}^{N_1} e^{-2\pi i q\lambda x} \right|^2 \, d\alpha
\]

\[
= \frac{4}{N_1} \int_0^1 F(\alpha) \sum_{|\lambda| \leq Nq^{-1}} (N_1 - |\lambda|) e^{-2\pi i \lambda q} \, d\alpha = 4 \int_0^1 F(\alpha) \Phi(\alpha) \, d\alpha.
\]

Hence we have

\[
\sigma(\mu) \leq 4 \int_0^1 F(\alpha) \Phi(\alpha) \, d\alpha = 4 \sum_{|\lambda| \leq Nq^{-1}} \left(1 - \frac{|\lambda|}{N_1}\right) c(q\lambda) \leq 4 \sum_{|\lambda| \leq Nq^{-1}} c(q\lambda).
\]

Substituting the inequalities obtained into (5), we obtain the assertion of the lemma. \(\square\)