On the Envelopes of Abelian Subgroups in Connected Lie Groups

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ABSTRACT. An Abelian subgroup $A$ in a Lie group $G$ is said to be regular if it belongs to a connected Abelian subgroup $C$ of the group $G$ (then $C$ is called an envelope of $A$). A strict envelope is a minimal element in the set of all envelopes of the subgroup $A$. We prove a series of assertions on the envelopes of Abelian subgroups. It is shown that the centralizer of a subgroup $A$ in $G$ is transitive on connected components of the space of all strict envelopes of $A$. We give an application of this result to the description of reductions of completely integrable equations on a torus to equations with constant coefficients.

Let $A$ be an Abelian subgroup in a connected Lie group $G$. The subgroup $A$ is said to be regular in $G$ if it is contained in a connected Abelian Lie subgroup $C$ of the group $G$ (then $C$ is called an envelope of $A$). If a subgroup $A$ is regular, then we shall mainly pay attention to minimal envelopes; we call them strict envelopes (since the envelopes are connected, it is clear that, in the ordinary set-theoretical sense, minimal envelopes exist). For the strict envelope $A^*$ of an Abelian subgroup $A$, its Lie algebra $L(A^*)$ is a minimal element of the set of Abelian Lie subalgebras of the Lie algebra $L(G)$ of the Lie group $G$ for which their image under the exponential mapping $\exp: L(G) \to G$ contains the original Abelian subgroup $A$. Denote by $E(A, G)$ the set of all strict envelopes of a subgroup $A$ in the Lie group $G$. Since a connected Lie subgroup is uniquely defined by its Lie algebra, this set is naturally identified with the set of Lie subalgebras of $L(G)$ that correspond to strict envelopes. Applying this identification, we can introduce a natural topology on $E(A, G)$ (see below). In this paper we present the description of all connected components of the topological space $E(A, G)$ by representing these components as orbits of the action of a connected Lie group that is determined by $G$ and by the subgroup $A$. As special cases, this description contains the results of A. I. Mal'tsev [1] (on the strict envelopes of the center $Z(G)$ of a connected Lie group $G$) and T. Nono [2] (on the one-parameter subgroups in a connected Lie group $G$ that pass through a chosen element $g \in G$). Moreover, we indicate an application to the study of completely integrable differential equations on the torus $\mathbb{T}^n$, which concerns their possible reduction to equations with constant coefficients (for details on such equations, see [3, 4]).

First, we briefly consider the existence problem for an envelope of a given Abelian subgroup $A$ in a connected Lie group $G$. Not every Abelian subgroup has such an envelope and thus is regular. For example, let $G = K$ be a connected simple compact Lie group. Then it is known that all finite (and even all compact) Abelian subgroups of $K$ are regular if and only if the Lie group $K$ is simply connected and its cohomology group with integral coefficients $H^*(K, \mathbb{Z})$ is torsion-free (that is, $H^*(K, \mathbb{Z})$ is a free Abelian group) [5, 6]. As is well known, a simple simply connected Lie group whose cohomology is torsion-free is isomorphic to $\text{SU}(n)$ or $\text{Sp}(n)$ for some $n \in \mathbb{N}$. Furthermore, for an arbitrary prime $p$, any subgroup of a simply connected compact simple Lie group $K$ that is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ is regular if and only if $H^*(K, \mathbb{Z})$ is $p$-torsion-free [5]. Similar results hold for finite Abelian subgroups of arbitrary connected Lie groups $G$ (because the problem is reduced to the consideration of finite Abelian subgroups in a maximal compact subgroup $K$ of the Lie group $G$) and also for Abelian subgroups $A$ of complex algebraic Lie groups all of whose elements $g \in A$ are semisimple (that is, completely reducible) [6]. As far as the regularity problem for more general Abelian subgroups is concerned, at present there are only some particular results in this direction [7].

If $A$ is a regular Abelian subgroup of a connected Lie group $G$, then we have $A \subseteq \exp(L(C))$, where $L(C)$ is a Lie subalgebra of $L(G)$ corresponding to an envelope $C$ of the subgroup $A$. Therefore,
an arbitrary regular Abelian subgroup is contained in the image of the exponential mapping for the Lie group $G$. If for a Lie group $G$ the mapping $\exp$ is not surjective, then in $G$ there exist Abelian subgroups that are not regular. More precisely, if we have $g \in G \setminus \exp(L(G))$, then the cyclic subgroup $A = \{g^k\}$ generated by the element $g$ is clearly not regular in $G$.

Furthermore, for a connected Lie group $G$ we consider its center $Z(G) = A$. For this Abelian subgroup $A$, the regularity problem was considered by A. I. Mal'tsev in [1]. In particular, he proved that if $G$ is solvable, then $Z(G)$ is a regular Abelian subgroup (in fact, in [1] the proof is given only for the case in which $G$ is simply connected, but in the editor’s note to the paper [1] reprinted in the selected works of A. I. Mal'tsev, it is shown how to prove the result for the non-simply-connected case). If $G$ is a connected semisimple Lie group, then its center is contained in a Cartan subgroup $H$ of a characteristic subgroup of the group $G$ (see [8], where by a characteristic subgroup of a semisimple Lie group $G$ a maximal Lie subgroup with compact embedding is meant). Since $H$ is connected, we see that for a semisimple Lie group $G$ its center is a regular subgroup. Now let $G$ be an arbitrary connected Lie group. Consider its Levi decomposition $G = S \cdot R$ (where $R$ is the radical and $S$ is a semisimple part, i.e., a Levi subgroup) and the natural epimorphism $\pi: G \to S' = G/R$ onto a semisimple Lie group $S'$ (which is locally isomorphic to $S$) with kernel $R$. Then $\pi(Z(G))$ is a central subgroup of $S'$; therefore, by the above discussion this subgroup is contained in a connected Abelian Lie subgroup $H$ of the Lie group $S'$. By setting $G' = \pi^{-1}(H)$, we obtain a connected solvable Lie subgroup of $G$, which contains $Z(G)$. Clearly, $Z(G) \subseteq Z(G')$; therefore, since $Z(G')$ is regular in $G'$ (see above), we see that $Z(G)$ is a regular subgroup of $G'$. Then $Z(G)$ is regular in $G$ as well, and this proves the following statement.

**Proposition 1.** The center $Z(G)$ of an arbitrary connected Lie group $G$ is a regular subgroup of $G$.

Note that in [1] a weaker assertion was proved: $Z(G)$ contains a regular subgroup of finite index. Below we show how to derive another result of the paper [1] on these subgroups from the results of our paper.

If $A$ is an Abelian subgroup of $G$ and $A$ is contained in a connected nilpotent subgroup, then, in general, $A$ can be nonregular in $G$. For example, consider the Lie group $F = N(3, \mathbb{R})$ of unipotent real matrices of order 3 and the lattice $\Gamma = N(3, \mathbb{Z})$ in it, which is the discrete subgroup consisting of integral unipotent matrices. Furthermore, consider the center $Z(F)$ (this group is isomorphic to $\mathbb{R}$) and its intersection with $\Gamma$; we can readily verify that this intersection is isomorphic to $\mathbb{Z}$. Now we set $G = F/Z(F) \cap \Gamma$ and $A = \Gamma/Z(F) \cap \Gamma$. It is easy to verify that the group $A$ is isomorphic to $\mathbb{Z}^2$, in particular, it is Abelian. The Lie group $G$ is nilpotent of nilpotency class 2. Here we can directly verify that any maximal connected Abelian Lie subgroup of $G$ is two-dimensional and isomorphic to $S^1 \times \mathbb{R}^1$ (where the Lie group $S^1$ is isomorphic to $SO(2)$). However, every such Abelian Lie subgroup cannot contain $A$ as a subgroup, because $A$ is a discrete subgroup of $G$ isomorphic to $\mathbb{Z}^2$. Thus, in the connected nilpotent Lie group $G$, we obtain an Abelian subgroup which is not contained in any connected Abelian Lie subgroup of the group $G$, that is, $A$ is not a regular subgroup. A peculiarity of the above example is that the corresponding connected nilpotent Lie group $G$ is not linear (i.e., it has no faithful finite-dimensional linear representation). The next assertion shows that this peculiarity is not accidental.

**Proposition 2.** Let $G$ be a connected nilpotent linear Lie group. Then an arbitrary Abelian subgroup of $G$ is regular.

**Proof.** In [1] it was proved that a connected solvable linear Lie group can be decomposed into the semidirect product of a connected Abelian compact subgroup (torus) and a simply connected normal subgroup. Therefore, for a Lie group $G$, we have the decomposition $G = T \cdot G_1$, where $T$ is a torus and $G_1$ is a simply connected nilpotent Lie group. Since in a nilpotent connected Lie group any connected compact subgroup is central (and therefore normal) in $G$, the semidirect product $T \cdot G_1$ is direct for the case under consideration. Thus, we obtain the decomposition of $G$ into the direct product $T \times G_1$. Denote by $\pi: T \times G_1 \to G_1$, the projection onto the direct factor. We set $A_1 = \pi(A)$; this is an Abelian subgroup in the simply connected nilpotent Lie group $G_1$. As is known, on the Lie group $G_1$ there is a natural structure of a real algebraic group. Denote by $\langle \pi(A) \rangle$ the algebraic closure of the subgroup $\langle \pi(A) \rangle$ in the algebraic group $G_1$; clearly, $\langle \pi(A) \rangle$ is a connected Abelian Lie group. We set $C = \pi^{-1}(\langle \pi(A) \rangle)$; it is a connected Abelian Lie subgroup of $G$ that contains the original Abelian subgroup $A$. Hence, the subgroup $A$ is regular in $G$. \[ \square \]