The Degrees of Polynomial Self-Mappings of \( \mathbb{C}^2 \)

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**Abstract.** Two results on the degrees of polynomial mappings \( \mathbb{C}^2 \to \mathbb{C}^2 \) are obtained.

**Key Words:** polynomial mappings, degree, Jacobian conjecture, Newton polygons, Puiseux series.

Let \( X_1, X_2 \) be the canonical coordinates in \( \mathbb{C}^2 \), and let \( \mathcal{C}[X_1, X_2] \leq m \) be the linear space of polynomials of degree \( \leq m \) of the variable \( X = (X_1, X_2) \). We choose an \( n = (n_1, n_2) \in \mathbb{N}^2 \) and set \( \tilde{V}_n = \mathcal{C}[X_1, X_2] \leq n_1 \times \mathcal{C}[X_1, X_2] \leq n_2 \). For \( F = (F_1, F_2) \in \tilde{V}_n \), consider the polynomial mapping (denoted by the same letter)

\[
F: \mathbb{C}^2 \to \mathbb{C}^2, \quad X = (X_1, X_2) \mapsto (F_1(X_1, X_2), F_2(X_1, X_2)) = F(X).
\]

Let \( J(F) \) be the Jacobian of \( F \). In the linear space \( \tilde{V}_n \) we consider the following closed submanifolds:

\[
D_i = \{ F \in \tilde{V}_n \mid \deg J(F) < i \}, \quad 0 \leq i \leq n_1 + n_2 - 2,
\]

\[
W_i = \{ F \in \tilde{V}_n \mid \dim F^{-1}(0) > 0, \text{ or } |F^{-1}(0)| < i, \text{ or } \deg F_1 < n_1 \text{ and } \deg F_2 < n_2 \}, \quad 0 \leq i \leq n_1 n_2.
\]

We address the following problem.

**Problem.** For what values of \( k \) and \( l \) does the inclusion \( D_k \subset W_l \) hold?

Note that the inclusion \( D_0 \subset W_1 \) for all \( n \) is equivalent to the two-dimensional Jacobian conjecture [1].

**Theorem 1.** For each \( k \geq 0 \) one has the inclusion \( D_k \subset W_{\min\{n_1, n_2\}(k+1)} \). In particular, if the Jacobian of a polynomial mapping \( F \) is identically equal to 1, then the degree of \( F \) does not exceed \( \min\{n_1, n_2\} \).

Let \( \mathcal{C}[X_1, X_2]_m \) be the linear space of homogeneous polynomials of degree \( m \) in the variables \( X = (X_1, X_2) \). A pair of elements \( F \in \tilde{V}_n, \ H' \in \mathcal{C}[X_1, X_2]_i \) is said to be generic if for \( i = 1 \) or \( i = 2 \) the restriction of the polynomial \( F_i(X) \) to the line \( H'(X) = 0 \) is a polynomial of degree \( n_i \).

**Theorem 2.** Let \( F \in \tilde{V}_n, \ H' \in \mathcal{C}[X_1, X_2] \) be a generic pair, and let \( |F^{-1}(0)| < \infty \). We introduce the linear subspaces \( K = K(F) \) and \( K_i = K_i(F, H') \subset \mathcal{C}[X_1, X_2] \leq n_1 + n_2 - 1 \) by setting

\[
K = \{ F_1 Q_2 + F_2 Q_1 \mid Q_i \in \mathcal{C}[X_1, X_2]_{n_i - 1} \},
\]

\[
K_0 = 0, \quad K_{i+1} = (K + H' K_i) \cap \mathcal{C}[X_1, X_2] \leq n_i + n_2 - 2.
\]

Then

\[
K_0 \subset \cdots \subset K_i \subset \cdots \subset \mathcal{C}[X_1, X_2] \leq n_1 + n_2 - 2,
\]

\[
2 \dim K_i \geq \dim K_{i-1} + \dim K_{i+1}, \quad i \geq 1, \quad |F^{-1}(0)| = n_1 n_2 - \dim K_{\infty}.
\]

We prove Theorem 1 in §1 and Theorem 2 in §§2–4.
§1. Let us recall some facts about Puiseux series.

A Puiseux series (in a neighborhood of infinity) is a series of the form

\[ \alpha(t) = \sum_{i \leq i_0} a_i(t^{1/d})^i \]

convergent for \(|t| > R\), where \(R > 0\). The degree \(\deg t \alpha(t)\) of the series \(\alpha(t)\) is the maximum number \(i/d\) such that \(a_i \neq 0\). We write \(\alpha \sim \beta\) if

\[ \beta(t) = \sum_{i \leq i_0} a_i\theta^i(t^{1/d})^i, \]

where \(\theta \in \mathbb{C}\) is a root of degree \(d\) of unity. The series \(\alpha(t)\) is called reduced if the greatest common divisor of the numbers \(\{i \mid a_i \neq 0\}\) is equal to 1. Suppose that \(\alpha(t)\) is a reduced Puiseux series. Then \(\alpha(t)\) specifies a \(d\)-valued analytic function

\[ \tilde{\alpha}: \{ t \in \mathbb{C} \mid |t| > R \} \to \mathbb{C}, \quad t \mapsto \alpha(t). \]

The number \(d\) is called the denominator of the series \(\alpha(t)\) (it will be denoted by \(\text{den}(\alpha)\)).

By \(\mathbb{C}\{t\}\) we denote the ring of Puiseux series. The differential operator \(d/dt\) maps this ring into itself. By \(\mathbb{C}[X_2][X_1]\) we denote the ring of polynomials in \(X_2\) with coefficients in the ring of Puiseux series of \(X_1\). The differential operators \(\partial/\partial X_i, i = 1, 2,\) map this ring into itself.

We say that an element \(G \in \mathbb{C}[X_1, X_2]\) is proper with respect to \(X_2\) if

\[ G(X_1, X_2) = G_0 X_2^p + G_1(X_1)X_2^{p-1} + \cdots + G_p(X_1), \]

where \(0 \neq G_0 \in \mathbb{C}\).

Let \(G\) be a polynomial in \(X_1, X_2\) proper with respect to \(X_2\). Then we can factorize \(G\) in the ring \(\mathbb{C}[X_2]\{X_1\}\). In other words, there exist reduced Puiseux series \(\alpha_1, \ldots, \alpha_m\) such that

\[ \text{den}(\alpha_1) + \cdots + \text{den}(\alpha_m) = \deg X_2 G, \quad G(X_1, X_2) = G_0 \prod_{1 \leq i \leq m} \left( \prod_{\alpha \sim \alpha_i} (X_2 - \alpha(X_1)) \right). \]

The Puiseux series \(\alpha_1, \ldots, \alpha_m\) are called the roots of \(G\) with respect to \(X_2\).

Let \(G\) be a polynomial in \(X_1, X_2\) proper with respect to \(X_2\). There is a well-known procedure for constructing the roots of \(G\) with respect to \(X_2\) (based on the use of Newton polygons). If we replace \(G\) by \(G + c\), where \(c \in \mathbb{C}\), then the roots (and sometimes even their number) will change. The following fact is a consequence of the procedure for constructing the roots.

**Lemma 1.** There exists a nonempty subset \(C = C(G) \subset \mathbb{C}\) such that if \(c \in C\), the Puiseux series \(\alpha_1, \ldots, \alpha_m\) are the roots of the polynomial \(G + c\) with respect to \(X_2\), and

\[ G(X_1, X_2 + \alpha_i(X_1)) + c = X_2 G_{i1}(X_1) + X_2^2 G_{i2}(X_1) + \cdots, \quad 1 \leq i \leq m, \]

then \(\deg_{X_1} G_{i1}(X_1) \geq 0, 1 \leq i \leq m\).

Let \(G_1\) and \(G_2\) be polynomials in \(X_1, X_2\) such that \(G_1\) is proper with respect to \(X_2\) and the system

\[
\begin{aligned}
G_1(X_1, X_2) &= 0, \\
G_2(X_1, X_2) &= 0
\end{aligned}
\]

has finitely many solutions. Let \(\alpha_1, \ldots, \alpha_m\) be the roots of \(G_1\) with respect to \(X_2\). Then the number of solutions (counting multiplicities) of system (1) is equal to

\[
\sum_{1 \leq i \leq m} \text{den}(\alpha_i) \deg_{X_1} G_2(X_1, \alpha_i(X_1))
\]

(Zeiten’s formula [2]).