Linearity of Metric Projections on Chebyshev Subspaces in $L_1$ and $C$

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ABSTRACT. Let $Y$ be a Chebyshev subspace of a Banach space $X$. Then the single-valued metric projection operator $P_Y : X \to Y$ taking each $x \in X$ to the nearest element $y \in Y$ is well defined. Let $M$ be an arbitrary set, and let $\mu$ be a $\sigma$-finite measure on some $\sigma$-algebra $\Sigma$ of subsets of $M$. We give a complete description of Chebyshev subspaces $Y \subset L_1(M, \Sigma, \mu)$ for which the operator $P_Y$ is linear (for the space $L_1[0, 1]$, this was done by Morris in 1980). We indicate a wide class of Chebyshev subspaces in $L_1(M, \Sigma, \mu)$, for which the operator $P_Y$ is nonlinear in general. We also prove that the operator $P_Y$, where $Y \subset C[K]$ is a nontrivial Chebyshev subspace and $K$ is a compactum, is linear if and only if the codimension of $Y$ in $C[K]$ is equal to 1.

KEY WORDS: Banach space, Chebyshev subspace, best approximation element, metric projection operator, quasiorthogonal set, linearity criterion.

Introduction

Let $X$ be a Banach space, and let $Y$ be a subspace (closed and linear). By

$$\rho(x, Y) = \inf\{\|x - y\| : y \in Y\}$$

we denote the distance from an element $x \in X$ to $Y$. Let

$$P_Y(x) = \{y \in Y : \|x - y\| = \rho(x, Y)\}$$

be the set of elements of $Y$ nearest to $x$.

A subspace $Y$ is called a Chebyshev subspace if for each $x \in X$ the set $P_Y(x)$ is a singleton. In other words, for each $x \in X$ there exists a unique best approximation element in $Y$. In general, the corresponding metric projection operator $P_Y : X \to Y$ that takes each $x \in X$ to the best approximation element $P_Y(x) \in Y$ is neither linear nor even continuous [1, 2].

The aim of the present paper is to describe Chebyshev subspaces $Y$ in $L_1$ and $C$ such that the operator $P_Y$ is linear (Theorems 1 and 3). Moreover, we indicate a rather wide class of Chebyshev subspaces $Y \subset L_1$ for which the operator $P_Y$ is nonlinear in general (Theorem 2).

Note that Theorem 1 is similar to a theorem due to Morris [3], who described Chebyshev subspaces with linear metric projection operators in the space $L_1[0, 1]$.

The main tool in our investigation is the notion of so-called quasiorthogonal set.

Definition. The set $Q(Y) = Q(Y, X)$ quasiorthogonal to a subspace $Y \subset X$ is the set of all elements $n \in X$ such that $P_Y(n) \ni 0$.

This object was first introduced by Cheney–Wulbert [2] and Singer [1] (the term quasiorthogonal set is due to E. P. Dolzhenko). If $X$ is a Hilbert space, then each element $n \in Q(Y)$ is orthogonal to the subspace $Y$ in the usual sense, and the set $Q(Y)$ itself is the orthogonal complement of $Y$.

Let us cite the main properties of quasiorthogonal sets.

The set quasiorthogonal to a subspace is always a closed two-sided cone: if $n \in Q(Y)$, then $\lambda n \in Q(Y)$ for every number $\lambda$. Obviously, in the simplest case $Y = X$ the set $Q(Y)$ is just the zero subspace, and conversely, $Q(\{0\}) = X$. 


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Lemma 1 [2]. A subspace $Y$ is a Chebyshev subspace if and only if the following two conditions are satisfied:

1) $Q(Y) + Y = X$;
2) $P_Y(n) = \{0\}$ for each $n \in Q(Y)$.

Moreover, 1) is equivalent to the existence of a best approximation element, and 2) is equivalent to its uniqueness.

Lemma 1 readily implies the following assertion.

Lemma 2 [1]. The operator $P_Y$ of metric projection on a Chebyshev subspace $Y$ is linear if and only if $Q(Y)$ is a linear subspace.

Let us indicate the following special case in which $P_Y$ is linear.

Lemma 3. If the linear codimension of a Chebyshev subspace $Y$ in $X$ is equal to 1, then $P_Y$ is a linear operator.

Proof. Indeed, we have $Q(Y) \neq \{0\}$ by Lemma 1. On the other hand, if $Q(Y)$ contains two linearly independent elements $p$ and $q$, then for some scalar $\lambda$ we have $y := p - \lambda q \in Y$, $y \neq 0$. It follows that $p - y = \lambda q \in Q(Y)$, and the element $p$ has at least two best approximation elements in $Y$, namely, $0$ and $y$. This is impossible, since $Y$ is a Chebyshev subspace. Thus $Q(Y)$ is a one-dimensional subspace, and the operator $P_Y$ is linear by Lemma 2. 

Condition 2) in Lemma 1 can be replaced by the assumption that the set quasiorthogonal to $Y$ is linear.

Lemma 4. If $Q(Y)$ is a subspace and $Q(Y) + Y = X$, then $Y$ is a Chebyshev subspace.

Proof. Indeed, suppose that the best approximation element for some $x \in X$ is not unique. Then $x = n_1 + y_1 = n_2 + y_2$, where $n_1, n_2 \in Q(Y)$, $y_1, y_2 \in Y$, and $y_1 \neq y_2$. It follows that $n_1 - n_2 = y_2 - y_1 \neq 0$. Since $Q(Y)$ is a subspace, we have $Q(Y) \ni n_1 - n_2 = y_2 - y_1 \in Y$, so that the set $Y \cap Q(Y)$ contains a nonzero element, which is impossible. The proof of Lemma 4 is complete. 

To conclude this section, let us prove an assertion that provides an immediate description of the set $Q(Y)$.

Let $Y^\perp = \{f \in X^* : f(y) = 0 \ \forall y \in Y\}$ be the annihilator of $Y$ in the dual space $X^*$.

Lemma 5. $Q(Y) = \{n \in X : \text{there exists a nonzero functional } f \in Y^\perp \text{ such that } f(n) = \|f\| \cdot \|n\|\}$.

Proof. Indeed, for each element $n$ and each $y \in Y$ we have

$$\|n - y\| \geq \frac{|f(n - y)|}{\|f\|} = \frac{|f(n)|}{\|f\|} = \|n\|,$$

so that $0 \in P_Y(n)$. Conversely, let $n \in Q(Y)$. Consider the one-dimensional subspace $(n)$ spanned by $n$ and the functional $f_0$ on $Y \oplus (n)$ given by the formula $f_0(y + \lambda n) = \lambda$. Since

$$|\lambda| = \frac{\|\lambda n\|}{\|n\|} \leq \frac{\|y + \lambda n\|}{\|n\|},$$

it follows that the norm of $f_0$ is equal to $1/\|n\|$. By the Hahn–Banach theorem, $f_0$ extends to a linear functional with the same norm on the entire space $X$; as a result, we obtain a functional $f \in Y^\perp$ such that $f(n) = 1 = \|n\| \cdot \|f\|$. 

Lemma 5 can be interpreted as follows: to obtain $Q(Y)$, we must take all norm-attaining functionals from $Y^\perp$ and then all elements of $X$ at which the norm is attained; $Q(Y)$ is just the set of all these elements.