Smoothness of Generalized Solutions of Boundary Value Problems for Certain Degenerate Nonlinear Ordinary Differential Equations

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ABSTRACT. The variational method is applied to the study of a boundary value problem of the first kind for a class of nonlinear ordinary differential equations of order $2r$ with strong degeneracy at the endpoints of the interval $(a, b)$. An inequality is obtained in which the norm of the solution $U$ of the problem under study in the sense of $W_{p,A}^r(a, b)$ is estimated from above by the norms of the given functions $\Phi(x)$ and $F(x)$.

KEY WORDS: variational method, boundary value problem of the first kind, nonlinear ordinary differential equation, Sobolev space, coerciveness inequality.

We study the differential properties of generalized solutions up to the boundary in relation to the differential properties of the coefficients of the equation (Theorem 3). We shall prove an inequality of the coerciveness type for the solution (see inequalities (42), (43)). Let $W_{p,A}^r = W_{p,A}^r(a, b)$ $(\Delta = (a, b))$ be the space of functions $f(x)$ for which the finite norm has a meaning (see [1, Chap. 10]):

$$
\|f\|_{W_{p,A}^r(\Delta)} = \|f\|_{p,\Delta} + \left\| \frac{f^{(r)}}{\rho^a} \right\|_{p,\Delta} < +\infty,
$$

where $r \geq 1$ is an integer, $\alpha$, $p$ $(1 < p < +\infty)$ are real numbers,

$$
\|f\|_{p,\Delta} = \|f^{(r)}\|_{p,\Delta}, \quad f^{(k)} = f^{(k)}(x) = \frac{d^k f(x)}{dx^k}, \quad k \leq r, \\
\rho = \rho(x) = \min\{x - a, b - x\} \quad \forall x \in \Delta = (a, b).
$$

Let

$$
0 < r + \alpha - \frac{1}{p} < r
$$

and $s_0$ is a natural number such that

$$
s_0 - 1 < r + \alpha - \frac{1}{p} \leq s_0, \quad \frac{r}{2} \leq s_0 \leq r.
$$

Under conditions (1) and (2), let us define the subspace (see [1, Chap. 10])

$$
W_0 = W_{p,A}^r = \{f \in W_{p,A}^r(a, b) : f^{(k)}(a) = f^{(k)}(b) = 0, \quad k = 0, 1, \ldots, s_0 - 1\}.
$$

Let us choose a function $\Phi(x) \in W_{p,A}^r(a, b)$ and by $W_\Phi$ denote the set of functions $f(x)$ belonging to $W_{p,A}^r(a, b)$ such that $f(x) - \Phi(x) \in W_0$. Under conditions (1) and (2), the following inequalities are valid (see [2-4]):

$$
\left\| \frac{f^{(k)}}{\rho^{a+r-k}} \right\|_{p,\Delta} \leq c_0 \left\| \frac{f^{(r)}}{\rho^a} \right\|_{p,\Delta} \quad \forall f \in W_0, \quad k \leq r, \\
\left\| \frac{f^{(k)}}{\rho^{a+r-k}} \right\|_{p,\Delta} \leq c_1 \|f\|_{W_{p,A}^r(\Delta)} \quad \forall f \in W_{p,A}^r(\Delta), \quad k \leq r
$$

$$
\beta_k = \left\{ \begin{array}{ll}
k, & k \geq s_0, \\
s_0, & k < s_0, \end{array} \right. \quad k \leq r.
$$
To each nonnegative integer \( k \) \((k \leq r)\) we assign a measurable function \( a_k(x) \) on \((a, b)\) such that
\[
\frac{c_2}{\rho^{p_k(r+a-\beta_k)}(x)} \leq a_k(x) \leq \frac{c_3}{\rho^{p_k(r+a-\beta_k)}(x)}, \quad 2 \leq p_k \leq p, \quad k = 0, 1, \ldots, r - 1, \quad p_r = p, \tag{5}
\]
where \( c_2 \) and \( c_3 \) are positive constants independent of \( x \in \Delta = (a, b) \). Let us consider the integral
\[
E(f) = \int_a^b \sum_{k \leq r} \frac{1}{\rho^{p_k}} a_k(x) |f^{(k)}(x)|^{p_k} dx,
\]
which is obviously finite for \( f \in W_r^{r, a}(a, b) \) under conditions (1), (2), and (5). Set \( H(f) = E(f) - (F, f) \), where \((F, f)\) is the inner product of the functions \( F(x) \) and \( f(x) \) by \( \Delta \) and the function \( F(x) \) satisfies
\[
\|\rho^{a+r-s\varphi} F\|_{q, \Delta} \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{6}
\]
Under conditions (1), (2), (5), and (6), the functional \( H(f) \) is bounded below in the class \( W_\varphi \). Set
\[
\inf_{f \in W_\varphi} H(f) = \lambda.
\]

Theorem 1. Under conditions (1), (2), (5), and (6), there exists a unique function \( U(x) \) belonging to \( W_\varphi \) satisfying the relations
\[
\min_{f \in W_\varphi} H(f) = H(U) = \lambda, \tag{7}
\]
\[
\int_a^b \left( \sum_{k \leq r} a_k(x) U^{(k)}(x) |U^{(k)}(x)|^{p_k - 2} U^{(k)}(x) v^{(k)}(x) - F(x) v(x) \right) dx = 0 \quad \forall f \in W_0. \tag{8}
\]

A function \( U \) belonging to \( W_\varphi \) and satisfying relation (8) for any \( v \in W_0 \), will be called the generalized solution of the boundary value problem

\[
LU = F(x), \quad x \in (a, b), \quad U \in W_\varphi, \quad \text{where} \quad LU = \sum_{k \leq r} (-1)^k (a_k(x) U^{(k)}(x) |U^{(k)}(x)|^{p_k - 2} U^{(k)}(x))^{(k)}.
\]

Theorem 2. The solution \( U \) of the variational problem (7) belonging to \( W_\varphi \) satisfies the inequality
\[
\|U\|_{W_{r, a}(\Delta)} \leq c_4 (\|\Phi\|_{W_{r, a}(\Delta)} + \|\rho^{a+r-s\varphi} F\|_{q, \Delta}), \tag{9}
\]
where
\[
\gamma_0 = \begin{cases} p, & \|\Phi\|_{W_{r, a}(\Delta)} \geq 1, \\ \min_k p_k, & \|\Phi\|_{W_{r, a}(\Delta)} \leq 1, \end{cases}
\]
c_4 is a constant independent of \( \Phi \) and \( F \).

Inequality (9) is proved by using identity (8).

Theorem 3. Suppose that conditions (1), (2), and (6) are satisfied, the functions \( a_k(x) \), where \( k \leq r/2, \quad k = r \), have continuous derivatives on the interval \((a, b)\) up to order \( k \) inclusive such that
\[
|a_k^{(s)}(x)| \leq \frac{c_5}{\rho^{p_k(r+a-\beta_k)+s}(x)}, \quad x \in (a, b), \quad s \leq k, \quad 2 \leq p_k \leq p, \quad k \leq r/2, \quad p_r = p, \tag{10}
\]