Cartan–Grauert Theorem for Tuboids With “Curvilinear” Edge

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Abstract. Tuboids are tube type domains with totally real edge that are asymptotically approximated near the edge points by local tubes over convex cones. For these domains we prove an analog of the Cartan–Grauert theorem on holomorphic convexity of domains in \( \mathbb{R}^n \subset \mathbb{C}^n \).

Key Words: domains of holomorphy, holomorphic convexity, tuboids.

H. Cartan [1] established the following result.

**Cartan Theorem.** Let \( M = \mathbb{R}^n = \mathbb{R}^n(x) + i\mathbb{R}^n(y) \) be the “real” subspace in \( \mathbb{C}^n = \mathbb{R}^n(x) + \mathbb{R}^n(y) \), and let \( \Omega \) be a domain in \( M \). Then \( \Omega \) has a fundamental system of neighborhoods in \( \mathbb{C}^n \) that are domains of holomorphy. In other words, for any open neighborhood \( U \) of \( \Omega \) in \( \mathbb{C}^n \) there is a domain of holomorphy \( D \) in \( \mathbb{C}^n \) such that \( \Omega \subset D \subset U \).

Grauert [2] extended this theorem to the case of arbitrary \( n \)-dimensional real-analytic manifolds \( M \).

In the present paper the Cartan–Grauert theorem is generalized to the case of tuboids. If \( M = \mathbb{R}^n(x) \subset \mathbb{C}^n \), then a *tuboid* is a domain in \( \mathbb{C}^n \) that asymptotically approaches a local tube of the form \( \Omega + iC_z \) near the points \( x \in M \), where \( C_z \) is an open cone in \( \mathbb{R}^n(y) \) and \( \Omega \) is an open neighborhood of \( x \) (the exact definition of tuboids is given in \$2.2\) below). In this case \( M \) is called the *edge of the tuboid* and the cone \( C_z \) is called its *profile* at \( x \); note that the profile \( C_z \) can depend on \( x \), i.e., change from point to point. The notion of tuboid was introduced by Bros and Iagolnitzer [3, 4]; they also obtained a generalization of the Cartan–Grauert theorem for tuboids with “rectilinear” edge, that is, for the case in which \( M \) is \( \mathbb{R}^n(x) \) (and also for the case in which \( M \) is an \( n \)-dimensional real-analytic totally real submanifold in \( \mathbb{C}^n \); the latter case reduces to the former).

In this paper we prove a generalization of the Cartan–Grauert theorem for tuboids with “curvilinear” edge, i.e., for the case in which \( M \) is the graph of a \( C^2 \)-smooth map \( H: \mathbb{R}^n(x) \rightarrow \mathbb{R}^n(y) \) with \( \|H'(x)\| < 1 \) for any \( x \in \mathbb{R}^n(x) \) (the exact statement of the theorem is presented in \$2.4\).

**§1. Topological preliminaries**

1.1. Admissible maps. Let \( X \) be a topological space. A topological structure on \( X \) can be determined with the help of maps \( F: X \rightarrow \Theta \), where \( \Theta \) is the 2-element set \( \{0, \bullet\} \), so that the subset \( \{x \in X : F(x) = 0\} \) is open in \( X \) and its complement \( \{x \in X : F(x) = \bullet\} \) is closed. Presetting a topology on \( X \) (i.e., a system of open or closed subsets of \( X \)) is equivalent in these terms to specifying a set \( \tau(X) \) of admissible maps \( F: X \rightarrow \Theta \). More precisely, by \( \land \) and \( \lor \) denote the standard operations of *conjunction* and *disjunction* on \( \Theta \). Identifying the element \( 0 \) with “True” and the element \( \bullet \) with “False,” we obtain the following table for the operations \( \land \) and \( \lor \):

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<th>( A \land B )</th>
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Original article submitted May 19, 1998.

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If $F$ and $G$ are maps from $X$ to $\Theta$, then the pointwise operations $F \land G$ and $F \lor G$ are defined. The operation $\land$ corresponds to the intersection of open subsets and the union of closed subsets, while the operation $\lor$ corresponds to the union of open subsets and the intersection of closed subsets.

**Definition 1.** A topology on $X$ is determined by a set $\tau(X)$ of admissible maps $F: X \to \Theta$ satisfying the following conditions:

a) the maps $F \equiv \bullet$ and $F \equiv \circ$ belong to $\tau(X);$

b) if $F$ and $G$ belong to $\tau(X)$, then $F \land G$ and $F \lor G$ belong to $\tau(X);$

c) for any family $\{F_\alpha\}$, $\alpha \in I$, of maps belonging to $\tau(X)$, we have

$$\bigvee_{\alpha \in I} F_\alpha \in \tau(X),$$

where the operation $\lor$ is defined as above by

$$\bigvee_{\alpha \in I} F_\alpha(x) = \begin{cases} \circ & \text{if there is } \alpha_0 \in I \text{ such that } F_{\alpha_0}(x) = \circ, \\
\bullet & \text{if } F_\alpha(x) = \bullet \text{ for any } \alpha \in I. \end{cases}$$

**Definition 2.** A base $B$ of topology $\tau(X)$ on $X$ is a subset $B$ of $\tau(X)$ satisfying the conditions:

a) the map $F \equiv \circ$ belongs to $B;$

b) if $F$ and $G$ belong to $B$, then there is a family $\{H_\alpha\}_{\alpha \in I}$ of maps $H_\alpha \in B$ such that we have $F \land G = \bigvee_{\alpha \in I} H_\alpha;$

c) the map $F$ belongs to $\tau(X)$ if and only if there is a family $\{F_\alpha\}_{\alpha \in I}$ of maps in $B$ such that $F = \bigvee_{\alpha \in I} F_\alpha.$

Any system $B$ of maps $F: X \to \Theta$ satisfying the above conditions defines a certain set of admissible maps $\tau(X)$ for which $B$ is the base. To obtain $\tau(X)$, it suffices to consider the minimal set of maps $F: X \to \Theta$ that contains the base $B$ and $F \equiv \bullet$ and is closed under disjunction.

1.2. Spaces of continuous maps.

**Definition 3.** A continuous map $f$ from a topological space $X$ to a topological space $Y$ is defined in "$\tau$-terms" as a map preserving $\tau$. More precisely, a map $f: X \to Y$ is continuous if and only if $F \circ f \in \tau(X)$ for any $F \in \tau(Y)$. Denote the space of the continuous maps from $X$ to $Y$ by $C(X, Y)$.

In particular, if the 2-element set $\Theta$ is endowed with the natural topology (in which the only open subset distinct from $\emptyset$ and $\Theta$ is $\{\circ\}$), then $\tau(X) = C(X, \emptyset)$ for any topological space $X$.

The set of all admissible maps $\tau(X)$ is itself a topological space. Its topology is determined by the set $\tau(X, \Theta)$ of admissible maps from $C(X, \Theta)$ to $\Theta$. The base of this topology consists of admissible maps of the form $\mathcal{F}_K: C(X, \Theta) \to \Theta$, where $K$ is an arbitrary compact subset of $X$ and the map $\mathcal{F}_K$ is defined as

$$\mathcal{F}_K(F) = \bigwedge_{K} F = \begin{cases} \circ & \text{if } F(x) = \circ \text{ for any } x \in K, \\
\bullet & \text{if } F(x_0) = \bullet \text{ for some } x_0 \in K. \end{cases}$$

The topology $\tau(X, \Theta)$ coincides with the compact open topology on $C(X, \Theta)$.

The quantifier $\bigwedge_{K} F$ that first appeared in the above formula is used throughout this paper and hence deserves separate consideration. For an arbitrary compact topological space $K$ and an admissible map $F \in \tau(K)$, we write

$$\bigwedge_{K} F = \begin{cases} \circ & \text{if } F(x) = \circ \text{ for any } x \in K, \\
\bullet & \text{if } F(x_0) = \bullet \text{ for some } x_0 \in K. \end{cases}$$

The quantifier $\bigwedge_{K} F$ possesses the following continuity property: if $\{F_k\}$ is a continuous family of admissible maps of a topological space $Y$ to $\Theta$, where the parameter $k$ runs over the compact set $K$, then the map $\bigwedge_{k \in K} F_k: Y \to \Theta$ is also admissible.