ON CONJUGACY OF HIGH-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. It is shown that the differential equation

\[ u^{(n)} = p(t)u, \]

where \( n \geq 2 \) and \( p : [a, b] \rightarrow \mathbb{R} \) is a summable function, is not conjugate in the segment \([a, b]\), if for some \( I \in \{1, \ldots, n-1\} \), \( \alpha \in [a, b] \), and \( \beta \in [a, b] \) the inequalities

\[ n \geq 2 + \frac{1}{2}(1 + (-1)^{n-1}), \quad (-1)^{n-I}p(t) \geq 0 \text{ for } t \in [a, b], \]

\[ \int_{a}^{\beta} \frac{(t-a)^{n-2}(b-t)^{n-2}|p(t)|dt \geq (n-l)^{l}\frac{(b-a)^{n-1}}{(b-\beta)(\alpha-a)} \]

hold.

Consider the differential equation

\[ u^{(n)} = p(t)u, \quad (1) \]

where \( n \geq 2 \), \( p \in L_{loc}(I) \), and \( I \subset \mathbb{R} \) is an interval.

The following definitions will be used below.

Equation (1) is said to be conjugate in \( I \) if there exists a nontrivial solution of this equation with at least \( n \) zeroes (each zero counted accordingly to its multiplicity) in \( I \).

Let \( I \in \{1, \ldots, n-1\} \). Equation (1) is said to be \((l, n-l)\) conjugate in \( I \) if there exists a nontrivial solution \( u \) of this equation satisfying

\[ u^{(l)}(t_1) = 0 \quad (i = 0, \ldots, l-1), \]

\[ u^{(l)}(t_2) = 0 \quad (i = 0, \ldots, n-l-1), \]

with \( t_1, t_2 \in I \) and \( t_1 < t_2 \).

Suppose first that \(-\infty < a < b < +\infty\) and \( p \in L([a, b]) \).

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Lemma. Let $a < \alpha < \beta < b$. Then the Green's function $G$ of the problem

\[ u^{(n)}(t) = 0 \quad \text{for } t \in [a, b], \]
\[ u^{(j)}(a) = 0 \quad (j = 0, \ldots, l - 1), \]
\[ u^{(j)}(b) = 0 \quad (j = 0, \ldots, n - l - 1), \]

satisfies the inequality

\[ (-1)^{n-l} G(t, s) > \frac{(b - \beta)(\alpha - a)(s - a)^{n-l-1}(b - s)^{l-1}(t - a)^{l-1}(b - t)^{n-l-1}}{(b - a)^{n-l}} \times \]
\[ \times \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \quad \text{for } a \leq t < s \leq \beta. \quad (2) \]

Proof. The function $G$ can be written in the form

\[ G(t, s) = \begin{cases} 
\sum_{i=n-l+1}^{n} (-1)^{i-1} x_i(t)x_{n-i+1}(s) & \text{for } a \leq s < t \leq b, \\
-\sum_{i=1}^{n-l} (-1)^{i-1} x_i(t)x_{n-i+1}(s) & \text{for } a \leq t \leq s \leq b, 
\end{cases} \]

where

\[ x_i(t) = \frac{(t - a)^{n-i}(b - t)^{i-1}}{(i-1)!(b - a)^{n-i}}. \]

It is easy to verify that for any fixed $s \in [a, b]$ the function \( \frac{(-1)^{n-l} G(t, s)}{x_{n-l+1}(t)x_{n+1}(s)} \) decreases on \([a, b]\) and the function \( \frac{(-1)^{n-l} G(t, s)}{x_{n-l+1}(t)x_{n+1}(s)} \) increases on \([a, b]\). Thus

\[ (-1)^{n-l} G(t, s) \geq (-1)^{n-l} G(s, s) \frac{x_{n-l}(t)}{x_{n-l}(s)} \quad \text{for } t \leq s. \quad (3) \]

Taking into account that

\[ (-1)^{n-l} G(s, s) = (-1)^{n-l} \sum_{i=1}^{n-l} (-1)^{i-1} x_i(s)x_{n-i+1}(s) = \]
\[ = \frac{(s - a)^{n-l}(b - s)^{n-l}}{(b - a)^{n-l}} \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \]
and

\[ \frac{x_{n-l}(t)}{x_{n-l}(s)} = \frac{(t - a)^l(b - s)^{n-l-1}}{(s - a)^l(b - s)^{n-l-1}}, \]