ORTHOGONAL RANDOM VECTORS AND THE HURWITZ-RADON-ECKMANN THEOREM

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Abstract. In several different aspects of algebra and topology the following problem is of interest: find the maximal number of unitary antisymmetric operators $U_i$ in $H = \mathbb{R}^n$ with the property $U_i U_j = -U_j U_i$ ($i \neq j$). The solution of this problem is given by the Hurwitz-Radon-Eckmann formula. We generalize this formula in two directions: all the operators $U_i$ must commute with a given arbitrary self-adjoint operator and $H$ can be infinite-dimensional. Our second main result deals with the conditions for almost sure orthogonality of two random vectors taking values in a finite or infinite-dimensional Hilbert space $H$. Finally, both results are used to get the formula for the maximal number of pairwise almost surely orthogonal random vectors in $H$ with the same covariance operator and each pair having a linear support in $H \oplus H$.

The paper is based on the results obtained jointly with N.P. Kandelaki (see [1,2,3]).

1. Introduction. Two kinds of results will be given in this paper. One is of stochastic nature and deals with random vectors taking values in a finite- or infinite-dimensional real Hilbert space $H$. The other is algebraic or functional-analytic, and deals with unitary operators in $H$. Our initial problem was to find conditions for almost sure orthogonality of random vectors with values in $H$. Then the question arose: what is the maximal number of pairwise almost surely orthogonal random vectors in $H$. The analysis of this question led us to a problem which is a natural extension of an old problem in linear algebra, finally solved in 1942. It can be called the Hurwitz-Radon-Eckmann (HRE) problem in recognition of the authors who made the crucial contribution in obtaining the final solution during the different stages of the investigation.

In Section 2 we give the formulation of this problem, provide its solution, and also give a brief enumeration of areas in which this problem is of primary interest. In Section 3 we give the solution of the generalized HRE problem.

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Section 4 is for the conditions of almost sure orthogonality of two random vectors in $H$. In Section 5 we give an analysis of these conditions. In Section 6 our initial problem of determining the maximal number of pairwise orthogonal random vectors is solved under some restrictions. These restrictions simplify the problem, so that the generalized HRE formula can provide the solution. Finally, in Section 7 we give the proofs of the theorems formulated in previous sections.

2. The Hurwitz-Radon-Eckmann Theorem. In this section we deal only with the finite-dimensional case: $H = \mathbb{R}^n$. To begin the formulation of the problem, we first recall that a linear operator $U : \mathbb{R}^n \to \mathbb{R}^n$ is called unitary (or orthogonal) if $U^* = U^{-1}$ (and hence it preserves the distances).

**HRE Problem.** Find the maximal number of unitary operators $U_i : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following conditions ($I$ is the identity operator):

$$U_i^2 = -I, \quad U_i U_j = -U_j U_i, \quad i \neq j. \quad (1)$$

The solution of this problem is the number $\rho(n) - 1$, where $\rho(n)$ is defined as follows: represent the number $n$ as a product of an odd number and a power of two, $n = (2a(n) + 1)2^{b(n)}$, and divide $b(n)$ by 4, $b(n) = c(n) + 4d(n)$, where $0 \leq c(n) \leq 3$. Then

$$\rho(n) = 2^{c(n)} + 8d(n). \quad (2)$$

The HRE problem is directly connected with the problem of orthogonal multiplication in vector spaces. A bilinear mapping $p : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is called an orthogonal multiplication if $\|p(x, y)\| = \|x\| \cdot \|y\|$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$. An orthogonal multiplication $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ exists if $k \leq \rho(n)$ and it can easily be constructed if we have $k - 1$ unitary operators satisfying conditions (1). Conversely, if we have an orthogonal multiplication $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ then we can easily construct $k - 1$ unitary operators with the properties (1). Of course, there can be different sets of orthogonal operators satisfying (1), and correspondingly, there can be different orthogonal multiplications $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$.

Formula (2) shows that we always have $\rho(n) \leq n$. The equality $\rho(n) = n$ holds only for $n = 1, 2, 4, 8$ and so there exists an inner multiplication in $\mathbb{R}^n$ only for those values of the dimension (and $\mathbb{R}^n$ becomes an algebra for those $n$). For $n = 1$, the corresponding algebra is the usual algebra of real numbers. For $n = 2, 4, 8$ and 8 we can choose the unitary operators in such a way that the corresponding algebras will, respectively, be the algebras of complex numbers, quaternions, and Kelly numbers. Properties like (1) arose also in the theory of representation of Clifford algebras.

The HRE problem first appeared in the investigation of the classical problem of computing the maximal number of linearly independent vector