Stability of Solutions of Nonlinear Systems With Unbounded Perturbations

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ABSTRACT. We study systems of differential equations with perturbations that are unbounded functions of time. We suggest a method for constructing Lyapunov functions to determine conditions under which the perturbations do not affect the asymptotic stability of the solutions.

KEY WORDS: unbounded perturbations, asymptotic stability, Lyapunov functions.

§1. In this paper we study the effect of nonstationary perturbations on the stability of solutions of systems of nonlinear differential equations. We assume that the perturbations are unbounded functions of time with a certain rate of growth.

Consider the system

\[ \dot{x}_s = f_s(X), \quad s = 1, \ldots, n. \]  

(1)

Here the \( f_s(X) \) are continuously differentiable \( \mu \)-th-order homogeneous functions, where \( \mu > 1 \) is a rational number with odd denominator. We assume that the zero solution of (1) is asymptotically stable.

Along with (1), we consider the perturbed system

\[ \dot{x}_s = f_s(X) + r_s(t, X), \quad s = 1, \ldots, n, \]  

(2)

where the functions \( r_s(t, X) \) are defined and continuous in the domain

\[ t \geq 0, \quad \|X\| < H, \quad H \geq 0, \]  

(3)

and satisfy conditions that guarantee the existence and uniqueness of the solutions of (2) as well as the continuous dependence of solutions on the initial data. Let Eq. (2) also have the zero solution. We study conditions under which perturbations do not affect the asymptotic stability of the zero solution.

Theorems on stability with respect to a nonlinear approximation were obtained by Malkin and Krasovskii [1, 2]. They assumed that the perturbations are bounded in the domain (3). It is known [2] that if in the domain (3) the functions \( r_s(t, X) \) satisfy the inequalities

\[ |r_s(t, X)| \leq D\|X\|^\sigma, \quad s = 1, \ldots, n, \]  

where \( D \) and \( \sigma \) are positive constants and \( \sigma > \mu \), then the zero solution of (2) is asymptotically stable.

In this paper we consider the case in which the perturbations are unbounded functions of time.

§2. Suppose that the inequalities

\[ |r_s(t, X)| \leq D(t + 1)^\alpha\|X\|^\sigma, \quad s = 1, \ldots, n, \]  

which restrict the rate of growth of the perturbations, are satisfied in the domain (3). Here \( D, \sigma, \) and \( \alpha \) are positive constants.

Let us find out by how much the order of the perturbation must exceed the order of the function \( f_s(X) \) so as to ensure that the asymptotic stability of the zero solution of (1) implies the asymptotic stability of the zero solution of (2).
Theorem 1. If the inequality
\[ \sigma > \alpha (\mu - 1) + \mu \] is satisfied, then the zero solution of (2) is asymptotically stable.

Proof. It is known [3, p. 115-123] that the asymptotic stability of the zero solution of (1) implies the existence of functions \( V(X) \) and \( W(X) \) with the following properties:

1) \( V(X) \) and \( W(X) \) are positive definite;

2) \( V(X) \) and \( W(X) \) are homogeneous functions of order \( m \) and \( m + \mu - 1 \), respectively, with \( m > 1 \);

3) \( V(X) \) is continuously differentiable;

4) the following relation is valid:
\[ \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} f_s(X) = -W(X). \]

In [2] it was proved that for \( t \geq t_0 \) the solutions \( X(t, X_0, t_0) \) of (1) satisfy the inequalities
\[ \|X(t, X_0, t_0)\|^{\mu-1} \leq (c_1 \|X_0\|^{1-\mu} + c_2 (t - t_0))^{-1}, \]
where \( c_1 \) and \( c_2 \) are positive constants.

Let us prove that there exist positive numbers \( \gamma, A, \) and \( T \) such that for an arbitrary solution \( X(t) = X(t, X_0, t_0) \) of Eq. (2) with initial data that satisfy the conditions
\[ t_0 \geq T, \quad \|X_0\|^{\mu-1} < \frac{\gamma}{t_0}, \] the inequality
\[ \|X(t)\|^{\mu-1} < \frac{A}{t} \]
holds for all \( t \geq t_0 \).

For the homogeneous functions \( V(X) \) and \( W(X) \), the inequalities
\[ a_1 \|X\|^m \leq V(X) \leq a_2 \|X\|^m, \quad \left| \frac{\partial V(X)}{\partial x_s} \right| \leq a_3 \|X\|^{m-1}, \quad s = 1, \ldots, n, \]
\[ b_1 \|X\|^{m+\mu-1} \leq W(X) \leq b_2 \|X\|^{m+\mu-1}, \]
where \( a_1, a_2, a_3, b_1, \) and \( b_2 \) are positive constants, hold for all \( X \in E^n \) (see [3]).

We choose the numbers \( \gamma, A, \) and \( T \) so as to satisfy the conditions
\[ 2ma_2 > (\mu - 1)b_1 \gamma, \quad \gamma < A, \quad (\mu - 1)b_1 a_1^{(\mu - 1)/m} A > 2ma_2^{1+(\mu - 1)/m}, \quad \frac{A}{T} < H^{\mu-1}, \]
\[ 2nDa_3 A^{(\sigma - \mu)/(\mu - 1)}(t + 1)^{\alpha - (\sigma - \mu)/(\mu - 1)} < b_1 \quad \text{for} \quad t \geq T. \]

We consider the solution \( X(t) \) of (2) with initial data that satisfy inequalities (5). Suppose that there exists a \( t_1 > t_0 \) such that
\[ \|X(t_1)\|^{\mu-1} = \frac{A}{t_1}, \]
and inequality (6) holds for \( t \in [t_0, t_1) \). Let us differentiate the function \( V(X) \). By (2), for \( t \in [t_0, t_1] \) we have
\[ \frac{dV(X(t))}{dt} = -W(X(t)) + \sum_{s=1}^{n} \frac{\partial V(X(t))}{\partial x_s} r_s(t, X(t)) \]
\[ \leq -b_1 \|X(t)\|^{m+\mu-1} + nDa_3(t + 1)^{\alpha} \|X(t)\|^{m+\sigma-1} < -\frac{b_1}{2} \|X(t)\|^{m+\mu-1}. \]