Limit Sets at Infinity for Liftings of Non-Self-Intersecting Curves on the Torus to the Plane

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ABSTRACT. In this paper Anosov's initial description of the sets mentioned in the title is completed. It is proved that there are four types of such sets and that all of these types are realizable.

KEY WORDS: infinite curves, torus, liftings, asymptotic direction, limit set.

In this paper we describe the sets of directions in which a lifting of a non-self-intersecting curve on the torus to the plane can accumulate at infinity.

Let us fix the terminology.

Unless otherwise stated, a curve on a surface is the image of a continuous mapping of a ray $\mathbb{R}_+ = [0, +\infty)$ to the surface. A lifting of a curve $\Gamma$ on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a curve $\tilde{\Gamma}$ on $\mathbb{R}^2$ that maps to $\Gamma$ under the natural projection $p: \mathbb{R}^2 \to T^2$. Any two liftings of a curve on the torus are transformed into each other under a translation by a vector of the integral lattice.

Remark 1. If a lifting of a non-self-intersecting curve on the torus goes to infinity as the parameter of the curve increases, then it does so along a definite direction. This theorem was stated by A. Weil [1] and proved by N. Markley [2]. Its analog for other two-dimensional closed oriented surfaces was stated by D. V. Anosov as a conjecture and proved by V. I. Pupko [3]. If a lifting oscillates, i.e., is unbounded and does not tend to infinity, then the above statement is generally false (see Theorem 1).

The problem of describing such sets of directions and its generalization to an arbitrary closed surface were set and partially solved by D. V. Anosov (see [4, 5]). These questions appeared as generalizations of a similar problem about the behavior of the liftings of the leaves of foliations with singularities on surfaces. In this paper the description of such sets of directions for the curves on the torus is completed.

Remark 2. No full description of such sets is known for curves on closed surfaces of genus greater than 1. In particular, the possible directions of passing to infinity for nonoscillating liftings (i.e., going to infinity as the parameter increases) for non-self-intersecting curves on these surfaces are unknown.

To state the main result, we introduce the following definition:

Definition. Suppose that $\Gamma$ is a curve in $\mathbb{R}^2$. A point $\alpha \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is called a limit point at infinity of $\Gamma$ if there is a sequence of points in $\Gamma$ such that they tend to infinity and the angles of inclination of their radius vectors tend to $\alpha$. The limit set at infinity of the curve $\Gamma$ is the set of its limit points at infinity. The limit set at infinity of a curve on the torus is the limit set at infinity of the lifting of this curve.

Remark 3. The limit set at infinity of a curve on the torus is closed and does not depend on the choice of the lifting.

The main result of the paper is the following theorem:

Theorem 1. The limit set at infinity of a non-self-intersecting curve on the torus can only be of one of the following types:

1) a point;
2) two diametrically opposed points;
3) a circular arc of length no greater than $\pi$;
4) the entire circle.

Conversely, any subset of type 1)–4) of the circle is the limit set at infinity of a non-self-intersecting curve on the torus.

Earlier D. V. Anosov [6] asserted that the limit set at infinity of a non-self-intersecting curve on the torus is either of one of types 1), 2), 4) or a circular arc or the union of two diametrically opposed circular arcs (the proof is unpublished). In [5] he proved that any subset of the circle of one of types 1), 2), 4) can be realized as the limit set at infinity of a non-self-intersecting curve on the torus. The same assertion for subsets of type 3) is also stated there without proof.

§1. Sketch of the proof of Theorem 1

The first assertion of Theorem 1 is implied by the following lemma:

Lemma 1. Suppose that $S \subset S^1$ is a closed subset such that one of the connected components of $S^1 \setminus S$ is a circular arc of length less than $\pi$. Then $S$ cannot be the limit set at infinity of a non-self-intersecting curve on the torus.

Lemma 1 is proved in §§1–3.

To prove the first assertion of Theorem 1, it remains to note that by Lemma 1 the complement of the limit set at infinity of a non-self-intersecting curve on the torus is either the empty set (type 4)) or a circular arc of length no less than $\pi$ (types 1) and 3) ) or a union of two diametrically opposed circular arcs of length $\pi$ (type 2)).

The second assertion of Theorem 1 follows from Anosov's result [5] on the realizability of the circle as the limit set and from the following lemma:

Lemma 2. Any circular arc of length no greater than $\pi$ can be realized as the limit set at infinity of a non-self-intersecting curve on the torus.

Lemma 2 is proved in §4.

§2. Sketch of the proof of Lemma 1

Suppose that $\alpha, \beta \in S^1$ are limit points at infinity of a curve $\Gamma$ on the torus and that they bound a circular arc of length less than $\pi$ containing no other limit points at infinity. We shall prove that in this case the curve $\Gamma$ is self-intersecting. This will imply Lemma 1.

Lemma 1 will be proved using the following evident remark:

Remark 4. Let $\Gamma$ be a curve on the torus, and let $\overline{\Gamma}$ be its lifting. The curve $\Gamma$ is non-self-intersecting if and only if so is $\overline{\Gamma}$ and, for any translation $g$ of the plane by a nonzero integral vector, the curves $\Gamma$ and $g\overline{\Gamma}$ do not intersect.

Lemma 1 is a consequence of the following lemma:

Lemma 3. Let $\Gamma$ be an unbounded non-self-intersecting curve in $\mathbb{R}^2$. Suppose that its limit points at infinity $\alpha$ and $\beta$ bound a circular arc $L$ of length less than $\pi$ containing no other limit points at infinity. Then there is a translation $g$ by a nonzero integral vector such that $\Gamma$ and $g\Gamma$ intersect.

§3. Proof of Lemma 3

Let $\Gamma, \alpha, \beta,$ and $L$ satisfy the conditions of Lemma 3. To prove the lemma, we construct a line segment $[A, B]$ with endpoints on $\Gamma$ that is parallel to and has the same length as an integral vector. Then this vector determines the desired translation $g$ of the plane.

Let $a$ and $b$ be the rays issuing from $O$ with angles of inclination $\alpha$ and $\beta$, respectively. By the condition of Lemma 3, the rays $a$ and $b$ are not in one line. There exists a line $l$ passing through two different vertices $A'$ and $B'$ of the integral lattice and intersecting both rays. Let us fix such a line $l$. We shall show that the line segment $[A, B]$ can be constructed so that it is parallel to the line segment $[A', B']$