GENERALIZED SIERPINSKI SETS

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ABSTRACT. The notion of a Sierpinski topological space is introduced and some properties of such spaces connected with Borel measures are considered.

We assume that all topological spaces $E$ to be considered below possess the following property: any singleton in $E$ is a Borel subset of $E$. In particular, all Hausdorff topological spaces possess this property.

We say that a topological space $E$ is a Luzin space if each $\sigma$-finite continuous, i.e., diffused, measure defined on the Borel $\sigma$-algebra of $E$ is identically zero.

We say that a topological space $E$ is a Sierpinski space if $E$ contains no Luzin spaces with cardinality equal to $\text{card}(E)$.

The classical Luzin set on the real line $R$ gives us a nontrivial example of an uncountable Luzin topological space (see, for example, [1]). Similarly, the classical Sierpinski set on $R$ gives us a nontrivial example of an uncountable Sierpinski topological space (see [1]). Therefore Luzin topological spaces (accordingly, Sierpinski topological spaces) may be considered as generalized Luzin sets (accordingly, as generalized Sierpinski sets). Some properties of Luzin and Sierpinski topological spaces are investigated in [2] and [3]. In this paper we investigate some other properties of Sierpinski topological spaces.

It is obvious that the cardinality of any Luzin topological space is strictly majorized by the first measurable (in the broad sense) cardinal number. The following simple proposition gives a characterization of measurable (in the broad sense) cardinal numbers in terms of Sierpinski topological spaces.

**Proposition 1.** Let $E$ be the main base set. Then the next two relations are equivalent:

1) $\text{card}(E)$ is a measurable (in the broad sense) cardinal number;
2) the topological space $(E, T)$ is a Sierpinski space for every topology $T$ on the set $E$.

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Proof. Indeed, if \( \text{card}(E) \) is a measurable (in the broad sense) cardinal number, then for each set \( X \subset E \) with \( \text{card}(X) = \text{card}(E) \) there exists a probability continuous measure defined on the family of all subsets of \( X \). Therefore for any topology \( T \) on the set \( E \) the space \((E, T)\) contains no Luzin spaces with cardinality equal to \( \text{card}(E) \). Thus we see that in this situation the topological space \((E, T)\) is a Sierpinski space. Conversely, let us assume that the topological space \((E, T)\) is a Sierpinski space for every topology \( T \) on the set \( E \). If we set \( T = \) a discrete topology on \( E \), then we immediately find that \( \text{card}(E) \) is a measurable (in the broad sense) cardinal number.

It is not difficult to verify that if \( E \) is a Sierpinski topological space and \( X \) is a subspace of \( E \) with \( \text{card}(X) = \text{card}(E) \), then \( X \) too is a Sierpinski topological space. \\

Proposition 2. Let \( E \) be a topological space and let
\[
E = \bigcup_{i \in I} E_i,
\]
where \((E_i)_{i \in I}\) is a finite family of Sierpinski subspaces of \( E \). Then the topological space \( E \) is a Sierpinski space. In particular, the topological sum of any finite family of Sierpinski spaces is also a Sierpinski space.

Proof. Let \( X \) be an arbitrary subspace of the space \( E \) with \( \text{card}(X) = \text{card}(E) \). Assume that \( X \) is a Luzin subspace of the space \( E \). Since the set of indices \( I \) is finite, there exists an index \( i \in I \) such that the equality \( \text{card}(E_i \cap X) = \text{card}(E_i) = \text{card}(E) \) is fulfilled.

Let us consider the set \( E_i \cap X \). This set, being a subset of the Luzin topological space \( X \), is also the Luzin space. At the same time, this set is a subspace of the topological space \( E_i \). Therefore we see that the Sierpinski topological space \( E_i \) contains the Luzin topological space \( E_i \cap X \) with the same cardinality, which is impossible. The obtained contradiction proves the proposition. 

We shall ascertain below that the result of Proposition 2, generally speaking, does not hold for topological sums of infinite families of Sierpinski spaces.

Let \( E \) be the main base set whose cardinality is not cofinal with the least infinite cardinal number \( \omega = \omega_0 \). We set \( T(E) = \{ X \subset E : \text{card}(E \setminus X) < \text{card}(E) \} \cup \{ \emptyset \} \). It is not difficult to verify that \( T(E) \) is a topology in \( E \) and the topological space \((E, T(E))\) is a Sierpinski space. It is further easy to ascertain that if \( \text{card}(E) = \omega_{\xi+1} \), then any subset of \((E, T(E))\) having cardinality \( \omega_{\xi} \) is discrete. Hence it follows that if \( \text{card}(E) = \omega_{\xi+1} \) and the cardinal number \( \omega_{\xi} \) is not measurable (in the broad sense), then any