Sequences of Maximal Terms and Central Exponents of Derivatives of Dirichlet Series

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ABSTRACT. For the Dirichlet series corresponding to a function \( F \) with positive exponents increasing to \( \infty \) and with abscissa of absolute convergence \( A \in (-\infty, +\infty) \), it is proved that the sequences \( (\mu(\sigma, F^{(m)})) \) of maximal terms and \( (\Lambda(\sigma, F^{(m)})) \) of central exponents are nondecreasing to \( \infty \) as \( m \to \infty \) for any given \( \sigma < A \), and

\[
\lim_{m \to \infty} \frac{\ln \mu(\sigma, F^{(m)})}{m \ln m} \leq 1 \quad \text{and} \quad \lim_{m \to \infty} \frac{\ln \Lambda(\sigma, F^{(m)})}{\ln m} \leq 1.
\]

Necessary and sufficient conditions for putting the equality sign and replacing \( \lim \) by \( \lim \) in these relations are given.

KEY WORDS: Dirichlet series, maximal term, central exponent.

§1. Introduction

Let \( 0 < \lambda_n \uparrow \infty \) and \( n \to \infty \), and let the Dirichlet series

\[
F(s) = \sum_{n=1}^{\infty} a_n \exp(s \lambda_n), \quad s = \sigma + it,
\]

have abscissa of absolute convergence \( A \in (-\infty, +\infty) \). We shall assume that all \( a_n \neq 0 \), so that the series (1) is not an exponential polynomial. For \( \sigma \in (-\infty, A) \), let \( \mu(\sigma, F) = \max\{|a_n| \exp(\sigma \lambda_n) : n \geq 1\} \) be the maximal term, let \( \nu(\sigma, F) = \max\{n \geq 1 : |a_n| \exp(\sigma \lambda_n) = \mu(\sigma, F)\} \) be the central index, and let \( \Lambda(\sigma, F) = \lambda_{\nu(\sigma, F)} \) be the central exponent of the series (1).

In the present paper we study the asymptotic behavior as \( m \to \infty \) of the sequences \( (\mu(\sigma, F^{(m)})) \) and \( (\Lambda(\sigma, F^{(m)})) \) for each given \( \sigma \in (-\infty, A) \), where \( F^{(m)} \) is the derivative of the function \( F \) of order \( m \in \mathbb{Z}_+ \).

The following theorem is valid.

**Theorem 1.** For each prescribed \( \sigma \in (-\infty, A) \), the sequences \( (\mu(\sigma, F^{(m)})) \) and \( (\Lambda(\sigma, F^{(m)})) \) are nondecreasing and tend to \( \infty \) as \( m \to \infty \).

The next theorem deals with the rate of growth of these sequences.

**Theorem 2.** For each prescribed \( \sigma \in (-\infty, A) \), we have

\[
\lim_{m \to \infty} \frac{\ln \mu(\sigma, F^{(m)})}{m \ln m} \leq 1 \tag{2}
\]

and

\[
\lim_{m \to \infty} \frac{\ln \Lambda(\sigma, F^{(m)})}{\ln m} \leq 1. \tag{3}
\]

The estimates (2) and (3) are sharp. This is a consequence of the following theorem.
Theorem 3. Let \( \sigma \in (-\infty, A) \). One has
\[
\lim_{m \to \infty} \frac{\ln \mu(\sigma, F^{(m)})}{m \ln m} = 1
\]
and
\[
\lim_{m \to \infty} \frac{\ln \Lambda(\sigma, F^{(m)})}{m} = 1
\]
if and only if
\[
\lim_{n \to \infty} \frac{1}{\ln \lambda_n} \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|a_n|} - \sigma \right) = 0.
\]

The natural question is: When can we replace \( \lim \) by \( \lim \) in (4) and (5)? The following theorem yields an answer to this question.

Theorem 4. Let \( \sigma \in (-\infty, A) \). One has
\[
\lim_{m \to \infty} \frac{\ln \mu(\sigma, F^{(m)})}{m \ln m} = 1
\]
and
\[
\lim_{m \to \infty} \frac{\ln \Lambda(\sigma, F^{(m)})}{m} = 1
\]
if and only if for any \( \varepsilon > 0 \) the following condition is satisfied:

1) for \( n \geq n_0(\varepsilon) \), the following inequality holds:
\[
\frac{1}{\ln \lambda_n} \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|a_n|} - \sigma \right) > -\varepsilon;
\]

2) there exists an increasing sequence \( (n_k) \) of natural numbers such that \( \ln \lambda_{n_{k+1}} \sim \ln \lambda_{n_k} \) as \( k \to \infty \) and
\[
\frac{1}{\ln \lambda_{n_k}} \ln \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|a_{n_k}|} - \sigma \right) < \varepsilon.
\]

A few corollaries of Theorems 3 and 4 are given at the end of the paper.

§2. Auxiliary results

Let \( \Omega \) be the class of positive functions \( \Phi \) unbounded on \((-\infty, +\infty)\) and such that the derivative \( \Phi' \) is continuous, positive, and increasing to \( +\infty \) on \((-\infty, +\infty)\). Let \( \varphi \) be the inverse of \( \Phi' \), and let \( \Psi(\sigma) = \sigma - \Phi(\sigma)/\Phi'(\sigma) \) be the function associated with \( \Phi \) in the sense of Newton.

Lemma 1 [1]. Suppose that \( \Phi \in \Omega \) and the Dirichlet series (1) has abscissa of absolute convergence \( A = +\infty \). For \( \ln \mu(\sigma, F) \leq \Phi(\sigma) \) to hold for all \( \sigma \geq \sigma_0 \), it is necessary and sufficient that \( \ln |a_n| \leq -\lambda_n \Psi(\varphi(\lambda_n)) \) for all \( n \geq n_0 \).

For arbitrary \( \varepsilon > 0 \), choose a function \( \Phi \in \Omega \) so that \( \Phi(\sigma) = (1 + \varepsilon)\sigma \ln \sigma \) for all sufficiently large \( \sigma \). Then \( \Phi'(\sigma) = (1 + \varepsilon)(\ln \sigma + 1) \), \( \Psi(\sigma) = \sigma/(\ln \sigma + 1) \), \( \varphi(t) = \exp(t/(1 + \varepsilon) - 1) \), \( t\Phi(\varphi(t)) = (1 + \varepsilon)\exp(t/(1 + \varepsilon) - 1) \). Therefore, by Lemma 1, for \( \ln \mu(\sigma, F) \leq (1 + \varepsilon)\sigma \ln \sigma \) to hold for all sufficiently large \( \sigma \), it is necessary and sufficient that \( \ln |a_n| \leq -(1 + \varepsilon)\exp(\lambda_n/(1 + \varepsilon) - 1) \) for all sufficiently large \( n \). This implies the following assertion.