Sufficient Conditions for the Existence of Feller Semigroups

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The problem of the existence of nonnegative contraction semigroups for one-dimensional diffusion processes with nonlocal boundary conditions was considered by W. Feller [1]. A. D. Ventsei' [2] proved that if a second-order elliptic differential operator is the infinitesimal generator of a Feller semigroup, then its domain consists of functions satisfying nonlocal boundary conditions. Sufficient conditions for the elliptic operator with such a domain to be the infinitesimal generator of a Feller semigroup were studied in [3-5] and other papers. In these papers it was assumed that the order of the nonlocal terms is lower than that of the boundary operator. The most difficult case in which the boundary and the nonlocal operators are of the same order was first considered by A. L. Skubachevskii [6-8] under the condition that the nonlocal terms decrease near the boundary. We obtain sufficient conditions for the existence of Feller semigroups assuming that the condition of decreasing of the nonlocal terms may be violated near some subset of the boundary.

§1. Let $X$ be a closed linear subspace in $C(\bar{Q})$, where $Q \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial Q \in C^\infty$, $n \geq 2$.

Definition 1. A semigroup $\{T_t\}_{t \geq 0}$ of linear closed operators acting on $X$ is called a Feller semigroup if $\|T_t\| \leq 1$ for $t \geq 0$ and the condition $f(x) \geq 0$ ($x \in \bar{Q}$) implies $(T_t)f(x) \geq 0$ ($x \in \bar{Q}$, $t \geq 0$).

Following [8], we consider the integro-differential operator $A$ of the form

$$ Au = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}x_j(x) + \sum_{i=1}^{n} a_i(x)u_{x_i}(x) + a(x)u(x) - \int_{Q} [u(x) - u(y)] m(x, dy) \quad (x \in Q) \quad (1) $$

with domain $\mathcal{D}(A) = \{ u \in C^2(Q) \cap C(\bar{Q}) : A u \in C(\bar{Q}) \}$. Here the $a_{ij}, a_i, a \in C^\infty(\mathbb{R}^n)$ are real-valued, $a_{ij} = a_{ji}$, and $m(x, \cdot)$ is a nonnegative Borel measure on $Q$. Let us set

$$ A^0 u = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}x_j(x) + \sum_{i=1}^{n} a_i(x)u_{x_i}(x) + a(x)u(x), \quad A^1 u = \int_{Q} u(y) m(x, dy). $$

Assume that the following conditions are satisfied:

1) $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j > 0$, $a(x) \leq 0$ for all $x \in \bar{Q}$ and $0 \neq \xi \in \mathbb{R}^n$;
2) the linear operator $A^1$ is bounded in the Hölder space $C^\sigma(\bar{Q})$, where $0 < \sigma < 1$.

Remark 1. It follows from condition 2) that $\gamma_m = \sup_{x \in \bar{Q}} m(x, \bar{Q}) < \infty$. Moreover, we have $m(x, \bar{Q}) \in C^\sigma(\bar{Q})$.

Consider the nonlocal boundary condition

$$ Bu = \gamma(x)u(x) + \int_{Q} [u(x) - u(y)] \mu(x, dy) = 0 \quad (x \in \partial Q), \quad (2) $$

where $\gamma(x) \geq 0$ and $\mu(x, \cdot)$ is a nonnegative Borel measure on $\bar{Q}$, $\gamma \in C^\sigma(\partial Q)$.
Let the following conditions be also satisfied:

3) \( \gamma(x) + \mu(x, \bar{Q}) > 0 \) \((x \in \partial \Omega)\);
4) \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \), \( \Gamma_1 \cap \Gamma_2 = \emptyset \), where \( \Gamma_2 = \{x \in \partial \Omega : \mu(x, \bar{Q}) = 0\} \) and the interior of \( \Gamma_2 \) (in the topology of the boundary) is not empty;
5) there exists a \( \Delta > 0 \) such that for any \( \delta, 0 < \delta < \Delta \), and any \( v \in C^\sigma(\bar{Q}) \) we have

\[
\int_{\Omega \setminus \Gamma_1^ \delta} v(y) \mu(x, dy) \in C^\sigma(\partial \Omega), \quad \int_{\Gamma_1^ \delta} v(y) \mu(x, dy) \in C^\sigma(\partial \Omega),
\]

\[
\left\| \int_{\Omega \setminus \Gamma_1^ \delta} v(y) \mu(x, dy) \right\|_{C^\sigma(\partial \Omega)} \leq c_1 \|v\|_{C^\sigma(\Gamma_1^ \delta)}, \quad \left\| \int_{\Gamma_1^ \delta} v(y) \mu(x, dy) \right\|_{C^\sigma(\partial \Omega)} \leq c_2(\delta) \|v\|_{C^\sigma(\Gamma_1^ \delta)},
\]

where \( \Gamma_1^ \delta = \{x \in \mathbb{R}^n : \rho(x, \Gamma_1) < \delta\} \), \( c_1 > 0 \) is independent of \( \delta \), and \( c_2(\delta) \to 0 \) as \( \delta \to 0 \), \( 0 < \sigma < 1 \).

**Remark 2.** Let \( v(y) \equiv 1 \). Then it follows from condition 5) that \( c_\mu = \sup_{x \in \partial \Omega} \mu(x, \bar{Q}) < \infty \), \( \mu(x, \bar{Q}) \in C^\sigma(\partial \Omega) \). Hence, the set \( \Gamma_2 \) is closed and \( \Gamma_1 \) is open in \( \partial \Omega \).

**Remark 3.** It follows from condition 5) that \( \sup_{x \in \partial \Omega} \mu(x, \bar{Q}) \leq c_2(\delta) \). Since \( c_2(\delta) \to 0 \) for \( \delta \to 0 \), we have \( \mu(x, \Gamma_1) = 0 \) \((x \in \partial \Omega)\).

Set \( C_B(\bar{Q}) = \{u \in C(\bar{Q}) : Bu = 0\} \). We define an operator \( A_B : C_B(\bar{Q}) \to C_B(\bar{Q}) \) by setting \( A_Bu = Au \) \((u \in D(A_B)), D(A_B) = \{u \in C^2(\Omega) \cap C_B(\bar{Q}) : Au \in C_B(\bar{Q})\}\).

**Theorem 1.** Let conditions 1)-5) be satisfied. Then \( A_B : C_B(\bar{Q}) \to C_B(\bar{Q}) \) is the infinitesimal generator of a Feller semigroup on \( C_B(\bar{Q}) \); this semigroup is uniquely determined by \( A_B \).

**Proof.** The proof is based on the Hille–Yosida theorem and on the extension of the separation method of nonlocal terms used in [8]. \( \square \)

**Example 1.** Consider the boundary condition

\[
u(x) - \sum_{s=1}^{N} b_s(x)u(\omega_s(x)) - \int_{\Omega} c(x, y)u(y) dy = 0 \quad (x \in \partial \Omega). \tag{3}
\]

Here \( b_s \in C^\sigma(\partial \Omega), c \in C^\sigma(\partial \Omega \times \Omega), \omega_s \) are continuously differentiable nondegenerate mappings of some neighborhood \( V \) of the boundary \( \partial \Omega \) onto \( \omega_s(V) \) such that \( \omega_s(\partial \Omega) \subset \bar{Q} \setminus \Gamma_1, \Omega \subset \bar{Q} \) is an open domain, \( \Gamma_2 \subset \partial \Omega \) is a closed subset with nonempty interior (in the topology of the boundary), \( \Gamma_1 = \partial \Omega \setminus \Gamma_2, b_s(x) = 0, \) and \( c(x, y) = 0 \) \((x \in \Gamma_2, y \in \Omega)\). In particular, one can consider the case in which \( \Omega = \bar{Q} \).

Suppose that the \( b_s(x) \) and \( c(x, y) \) are nonnegative and

\[
\sum_{s=1}^{N} b_s(x) + \int_{\Omega} c(x, y) dy \leq 1 \quad (x \in \partial \Omega).
\]

Then (3) can be rewritten in the form (2), and conditions 3)-5) are satisfied.

Indeed, let

\[
\gamma(x) = 1 - \sum_{s=1}^{N} b_s(x) - \int_{\Omega} c(x, y) dy, \quad \mu(x, G) = \sum_{s=1}^{N} b_s(x) + \int_{G \cap \Omega} c(x, y) dy
\]

for any Borel measure \( G \subset \bar{Q} \), where \( J(x, G) = \{s : \omega_s(x) \subset G\} \). Then the nonlocal condition (3) acquires the form (2), that is, for \( x \in \partial \Omega \),

\[
\left(1 - \sum_{s=1}^{N} b_s(x) - \int_{\Omega} c(x, y) dy\right)u(x) + \sum_{s=1}^{N} \left(u(x) - u(\omega_s(x))\right)b_s(x) + \int_{\Omega} (u(x) - u(y))c(x, y) dy = 0.
\]

Since \( \gamma(x) + \mu(x, \bar{Q}) \equiv 1 \) \((x \in \partial \Omega)\), condition 3) is satisfied. Let

\[
\Delta = \min_{1 \leq s \leq N} \rho(\omega_s(\Gamma_1), \Gamma_1).
\]

Then conditions 4) and 5) are satisfied.