Efficiency of a Corrugated Shell as a Radial Spring

Efficiency of material utilization is established from formulas for spring efficiency, elastic properties and stress of corrugated shells

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ABSTRACT—This paper deals with the problem of radial support of prismatical rods. The radial expansion of the rods and the limits of radial pressure acting on them are prescribed, so that an upper limit of the spring rate is imposed. The weight of the radial support must be kept as low as possible; hence, its efficiency of material utilization should be high. Specifically, the efficiency of a corrugated cylindrical shell employed as a radial-spring system is investigated. As a preliminary convenience, general formulas are set up to describe the efficiency of material utilization in springs. Then, the elastic properties and the stresses of corrugated shells with various shapes of convolutes are established. Finally, from the formulas for the spring efficiency, the elastic properties, and the stress of the corrugated shells, their efficiency of material utilization as springs is established.

Introduction

The information to be presented here was developed during the investigation of various methods of support for the core of an air-cooled nuclear reactor. Thermal expansion of the reactor core requires an elastic, radial support, and over-all weight considerations dictate a light structure. For this reason, the efficiency of material utilization for various support methods was investigated. One of these support methods, the corrugated shell, is analyzed in this paper.

List of Symbols

- $x, y$ = rectangular coordinates
- $\sigma, \tau$ = normal and shear stresses
- $\nu$ = Poisson's ratio
- $E, G$ = Young's modulus and shear modulus
- $U_s, U_e$ = strain energy and external energy of spring
- $\delta$ = deflection of extension spring
- $\phi$ = deflection of torsion spring
- $P$ = external force acting on spring
- $M$ = external moment acting on spring
- $k_e, k_t$ = spring constants of extension and torsion springs
- $V$ = volume of spring material
- $\psi$ = over-all efficiency of material utilization of spring
- $\psi_1$ = cross-section efficiency of spring
- $\psi_2$ = configuration efficiency of spring
- $R$ = radius of circumscribing circle of polygon
- $n$ = number of polygon sides
- $c$ = height of convolute over polygon side
- $h = \frac{c}{n}$ = half-length of polygon side
- $z = \frac{h}{c}$ = steepness factor of convolute
- $t$ = thickness of convolute wall
- $I$ = moment of inertia
- $T$ = pair of external forces acting on individual convolute
- $l$ = developed length of convolute

Efficiency of Material Utilization in Springs

In some cases, it is desirable to calculate the efficiency of material utilization for a proposed type of spring before specific designs and design calculations are made. The external parameters are spring rate and maximum spring force; the spring parameters are material, cross section, and configuration. In the following, these parameters are correlated to permit quick estimates of spring efficiency, or of volume of material needed, for springs proposed to fulfill given requirements.

The strain energy of a spring with all elements in tension or compression is:

$$U_s = \int \sigma \mathrm{d}V/2E = \psi V \sigma_{\text{max}}^2/2E$$

(1a)

where $\sigma_{\text{max}}$ is the maximum direct stress.

Similarly, the strain energy for a spring with all
elements in shear is:

\[ U_i = \int \sigma \, dV = 2G \int \tau \, dV \]  

where \( \tau_{\text{max}} \) is the maximum shear stress.

In the case of uniformly stressed springs, where \( \sigma = \sigma_{\text{max}} \) or \( \tau = \tau_{\text{max}} \) at all points, eqs (1a) and (1b) immediately lead to \( \Psi = 1 \). Such springs have the minimum volume, \( V_{\text{min}} \), for a given requirement of external work, \( U_i \), and \( V_{\text{min}} \) can be calculated from the relationships

\[ U_i = V_{\text{min}} \frac{\sigma_{\text{max}}}{2} \]  

or

\[ U_i = V_{\text{min}} \frac{\tau_{\text{max}}}{2} \]  

The efficiency of material utilization of a spring can now be defined as:

\[ \psi = \frac{\int \sigma \, dV}{\sigma_{\text{max}} V} = \frac{V_{\text{min}}}{V} \]  

or

\[ \psi = \frac{\int \tau \, dV}{\tau_{\text{max}} V} = \frac{V_{\text{min}}}{V} \]  

Thus, for a given spring, such as a proving ring—where the volume, the allowable force for a prescribed maximum stress, and the spring rate can be calculated from available formulas—the stress integration of eqs (1) or (3) need not be carried out to calculate spring efficiency; it is merely necessary to calculate the minimum volume of a spring capable of the same amount of external work, and to divide this minimum volume by the true volume.

The external work of an extension spring is:

\[ U_i = P_{\text{max}} \phi / 2 = P_{\text{max}}^2 / 2k_i \]  

and of a torsion spring:

\[ U_i = M_{\text{max}} \phi / 2 = M_{\text{max}}^2 / 2k_i \]  

Then, for instance, from eqs (2a) and (4a), the minimum volume of an extension spring is calculated as:

\[ V_{\text{min}} = \frac{P_{\text{max}}^2}{\sigma_{\text{max}} k_i} \]  

and the spring efficiency is obtained from eq (3a) and (5) as:

\[ \psi = \frac{P_{\text{max}}^2}{\sigma_{\text{max}} k_i} \frac{E}{2} \]  

For springs with similar, and similarly loaded, cross sections over their entire volume, the efficiency can be expressed as:

\[ \psi = \psi_1 \psi_2 \]  

TABLE 1—CROSS-SECTION EFFICIENCY OF SPRING, \( \psi \)

<table>
<thead>
<tr>
<th>Cross Section and Comments</th>
<th>Loaded in Tension or Compressive</th>
<th>Loaded in Tension ( \sigma_{\text{max}} ) is prescribed</th>
<th>Loaded in Torsion ( \tau_{\text{max}} ) is prescribed</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Case</td>
<td>1</td>
<td>0.4/\pi^2</td>
<td>0.2/\pi^2</td>
</tr>
<tr>
<td>Rectangular</td>
<td>1</td>
<td>0.332</td>
<td></td>
</tr>
<tr>
<td>Square</td>
<td>1</td>
<td>0.330</td>
<td>0.307</td>
</tr>
<tr>
<td>Thin-Wall Circle</td>
<td>1</td>
<td>0.500</td>
<td>1.000</td>
</tr>
<tr>
<td>Full Circle</td>
<td>1</td>
<td>0.250</td>
<td>0.500</td>
</tr>
</tbody>
</table>

Note: \( r = \) Radius of gyration; \( d = \) Distance from neutral axis to extreme fiber

where \( \psi_1 \) and \( \psi_2 \) are defined in a fashion analogous to the first expression for \( \psi \) in eqs (3), except that the integral for the cross-section efficiency, \( \psi_1 \), is taken over a cross-sectional area, and the integral for the configuration efficiency, \( \psi_2 \), is taken along an extreme fiber. Table 1 shows the cross-section efficiency and Table 2 the configuration efficiency for various springs.

Analysis of Corrugated Shell

Ideally, the corrugation shape as presented in rectangular coordinates should be "wrapped around" the mean circumference of the shell (for example, see Fig. 1). However, such a configuration complicates the analysis. For rapid analysis, a simplification has to be made. One has two choices—an approximate mathematical analysis of the accurate configuration, or a mathematically rigorous analysis of an approximate configuration. The second choice has been made, and it appears to yield satisfactorily accurate results through relatively simple formulas.

The simplification chosen uses a polygon as the basic configuration, and individual convolutes extend (in rectangular coordinates) from the polygon sides (see Fig. 2). Mutually compatible deformation of the corrugations requires that the tangents to the corrugations at the corners of the polygon remain parallel to each other. For corrugations where each half is antisymmetric with respect to its midpoint (S in Fig. 3), the condition of compatibility further requires that the lines of equal and opposite external forces, \( T \), pass through the midpoints. This type of corrugation will be considered here. For this configuration, the change in