Moiré Fringes as Parametric Curves

Parametric properties of families of curves are used by authors to express in a simple manner several fundamental properties of moiré fringes by A. J. Durelli and V. J. Parks

ABSTRACT—Advantage is taken in this paper of the parametric properties of families of curves to express in a simple manner several fundamental properties of moiré fringes. Attention is called, in particular, to the necessary limitations on the angle of rotation of two gratings, and on the magnitude of their difference in pitch, to obtain an easily interpretable moiré-fringe pattern.

Introduction

Moiré patterns are interference patterns produced by the superposition of two gratings, one on the other. It has been shown\(^1\) \(^2\) that a moiré pattern can be described analytically, in a parametric manner, if the analytical expressions for the two grating lines that produce the moiré pattern are known. In general, in the application of moiré to the determination of displacement fields, only one grating, the master grating, is expressible in simple analytic terms. Both the grating on the specimen and the moiré-fringe pattern exhibit some complex shapes associated with the deformations of the specimen. When this is the case, the relation between the lines of the two gratings and the associated moiré fringes can be obtained using parametric properties of families of curves by means of geometric, mechanical, or numerical procedures. However, if the strain analysis is limited to a small area of homogeneous strain, or to some simple field of displacement which has a known analytic expression, then the analytic method of determining the parameters of the resulting curves can be a very useful description of the moiré pattern in that small area or in the simple field.

In this paper, the parametric approach will be applied, first, to the study of the case of the superposition of two sets of equidistant straight lines of slightly different pitch, the superposition taking place in such a manner that the lines of one set make a small angle with the lines of the second set.

This example can be used to describe the deformation which would occur in a sufficiently small area of a strained body. The analytic expressions obtained give the deformations of the specimen grating in terms of measurements of the moiré pattern.

Second, the parametric approach is applied to the study of the case of the superposition of equidistant straight lines on equiangular straight lines. This example corresponds to the deformations produced in a beam subjected to bending. It illustrates the star, a common figure in moiré patterns. Both of these examples follow the approach of Oster, Wasserman and Zwerling\(^1\).

The approach is then extended to consider the case of two superposed sets of equidistant straight lines: first, where there is a large angle between the two sets of lines and, second, where there is a great variation in pitch between the two sets of lines. Both of these cases give moiré patterns which must be interpreted differently from the first example in order to determine deformations. The superposition of a radial- and straight-line grating is finally used to illustrate the variation in pattern that occurs due to large variations in pitch.

Superposition of Two Sets of Equidistant Straight Lines

Consider the pattern (Fig. 1) produced by a set of parallel opaque lines alternated with equiwidth transparent spaces superposed on a second similar set of opaque lines and spaces still of equal width but of slightly larger width than the first set. Assume that the direction of the lines in the first set differs somewhat from the direction of the lines in the second set.

In a Cartesian coordinate system is applied with directions perpendicular and parallel to the lines of the first set as shown in the figure, the family of lines in the first set can be described analytically by

\[ x = lp \]  \(1\)

where \( p \) is the width between lines (the pitch of the
grating; and \( l \) is the order number of each line or parameter, assuming that the line passing through the origin is given the number zero, and the lines away from the origin are ordered 1, 2, 3 in the positive \( x \)-direction and \(-1, -2, -3\) in the negative \( x \)-direction.

The family of lines in the second set can be described analytically by

\[
x = \frac{mp_1}{\cos \theta} + y \tan \theta
\]

where \( p_1 \) is the pitch of the second grating and \( \theta \) is the angle between the two sets of grating lines. Since we are interested in the interference of the two gratings, moiré fringes, or the locations where the dark areas intersect the light areas, the Cartesian coordinates have been chosen with the origin on a transparent space in the second grating. Then the parameter \( m \) will refer to the transparent spaces, with zero order at the origin, \( m = 1, 2, 3 \ldots \) to the left and \( m = -1, -2, -3 \ldots \) to the right in the figure.

The origin has been chosen at a point of interference and the orders chosen so that at the origin \( l = m = 0 \). Note that the interference at the origin is on the same moiré fringe as the interference occurring at \( l = m = 1 \), and that the interferences occurring at \( l = m = 2 \) and at \( l = m = 3 \) are also on that same fringe. The general relation of \( l \) to \( m \) for the moiré fringe through the origin is then

\[
l = m
\]

Consider next the interference just above the origin. Here the interference occurs at \( l = 0, m = -1 \). Again, notice that the fringe continues through the points \( l = 1, m = 0; l = 2, m = 1; \) and \( l = 3, m = 2 \), at which interference also takes place. The fringe is described in general by \( l = m = 1 \). Similar considerations show that the moiré fringes in general are described by

\[
l - m = n
\]

where \( n \) is an integer and the parameter of the moiré fringes.

Both \( l \) and \( m \) have been introduced as integers; however, note that the moiré fringe described by \( n \) is located by any value of \( l \) and \( m \) (integer or non-integer) which gives an integral value of \( n \). As such, the expression (3) is continuous over the field and represents the moiré fringes as a whole, not just the intersections of opaque and light areas in the field. In fact, the value of \( n \) can also be interpreted as a non-integer to give a continuous family in which the moiré fringes are selected members.

Substituting eq (1) and eq (2) into eq (3) gives the equation of the moiré fringe in terms of \( x, y \) and \( n \).

\[
\frac{x}{p} - \frac{y \cos \theta}{p_1} = n
\]

or in the form

\[
Ax + By + C = 0
\]

\[
x(p_1 - p \cos \theta) + yp \sin \theta - np_1 = 0
\]

The space between fringe lines can be computed using analytic geometry. Thus, the distance from the origin to the fringe of order \( n = 1 \) is

\[
\delta = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{pp_1}{\sqrt{p^2 \sin^2 \theta + (p_1 - p \cos \theta)^2}}
\]

This is the spacing of fringes throughout the pattern. The slope of the fringes measured from the master grating is the negative of the coefficient of \( y \) divided by the coefficient of \( x \);

\[
\tan \phi = -\frac{p \sin \theta}{p_1 - p \cos \theta}
\]

The position of the second (or specimen) grating is described completely from the spacing \( \delta \), angle \( \phi \) of the moiré grating, and the pitch of the first (or master) grating \( (p) \).

\[
p_1 = \frac{\delta}{\sqrt{1 + \left(\frac{\delta}{p}\right)^2 + 2 \left(\frac{\delta}{p}\right) \cos \phi}}
\]

\[
\theta = \text{arc tan} \left(\frac{\sin \phi}{p + \cos \phi}\right)
\]

These equations have been developed from a different point of view elsewhere and nomographs.