A Multilevel Treatment of Moiré Fringe Data Using Finite Elements

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ABSTRACT—Finite element procedures are applied to the modeling, analysis and visualization of experimental moiré data. Smoothing elements are introduced and evaluated with respect to data sparseness and error. A one-dimensional smoothing element is uniquely coupled with the method of principal curves to extract moiré fringe centers. A two-dimensional smoothing element is then used to produce a full-field representation given the fringe locations. The moiré technique is applied to the four-point bend experiment, and the surface-modeling technique is used to obtain displacement and gradient (strain) information.

KEY WORDS—surface modeling, moiré, principal curves, smoothing elements, fringe analysis

Finite element methodologies are herein examined and proposed as instruments to facilitate accurate gradient (strain) estimates given discrete data of unknown error and spatial distribution. The finite element method, FEM, normally features a piecewise approximation to an underlying function with weak interelement continuity control. The latter characteristic would seem to disqualify the procedure from consideration given the stated gradient requirement. In fact, this deficiency is well recognized and often used to monitor the accuracy and convergence of the technique. 1,2 Higher-order interelement continuity considerations are encountered in the derivation of certain elements such as those used to model plate behavior. Hinton and Irons3 utilized this type of formulation in their investigation of the FEM as a global smoothing technique.

Other researchers have explored various spline and spline-like procedures. Rowlands, Winters and Jensen4 examined spline regression. Segalman, Woyak and Rowlands5 introduced a finite element formulation that included least-squared smoothing technique. Morton et al.6-15 introduced a hybrid model that encompasses independent data types, experimental information and a standard FEM formulation.

This investigation extends the development of the 9 degree-of-freedom, (DoF), triangular plate element discussed in Tessler and Hughes11,12 and later explored as a smoothing candidate in Tessler et al.13-15. A curvature constraint term is added to the functional statement and the choice of weighting constant discussed. This formulation supports geometric modeling and provides maximum efficiency and local stability. The smoothing application allows but does not require a specification of values for both the function and its gradient and optionally provides for the simultaneous smoothing of multiple data sets.

The analysis and treatment of a moiré image begins with the determination of the fringe center locations. In this investigation, the cloud of data representing each fringe is uniquely examined by the principal curves method, PCM, introduced by Hastie.16,17 The method reformulates a multidimensional problem in terms of one-dimensional solutions. The robust, local regression smoothing technique described by Cleveland18 is adopted and independently applied to each of the one-dimensional problems. This step is uniquely implemented through a continuous one-dimensional version of the proposed smoothing element. The PCM/FEM procedure provides a mathematically based feature extraction tool, which can be used to determine moiré fringe center locations. By avoiding issues regarding sampling selection and minimizing the requirement for manual intervention, this initial treatment of fringe information establishes an independent and beneficial alternative to those outlined in other procedures.

As stated earlier, the usual sequence of operations in the mathematical treatment of experimental moiré images begins with an estimate of the fringe centers followed by a two-dimensional coupling of the fringe data. This report documents a unique approach to fringe center extraction, adds a curvature constraint term to Tessler's two-dimensional smoothing element and utilizes a one-dimensional version of the element in the fringe extraction algorithm. Because of this interrelationship, the ideas are presented in the following sequence: (a) a review and enhancement of Tessler's two-dimensional smoothing element, (b) an assessment of the new curvature constraint term, (c) the introduction of the unique fringe center procedure and (d) the hypothetical and experimental verification of the multilevel treatment of fringe images.
Two-Dimensional Smoothing Element

Consider the construction of a suitable approximation to the smooth function \( U(x, y) \) defined in region \( \Omega \), contained in Euclidean 2D space, given the elements of an arbitrary representative data set, \( u_i \) (e.g., discrete moiré fringe image information). Introduce the following quadratic functional

\[
\Phi(U) = \frac{1}{2N_d} \sum_{i=1}^{N_d} \omega_i \left[ U(x_i, y_i) - u_i \right]^2
\]

\[
+ \frac{\alpha}{2} \int_\Omega \left[ (U_x - \theta_x)^2 + (U_y - \theta_y)^2 \right] d\Omega
\]

\[
+ \frac{\beta}{2} \int_\Omega \left[ U_{xx}^2 + 2U_{xy}^2 + U_{yy}^2 \right] d\Omega,
\]

where \( N_d \) is the total number of data points, \( \theta_x \) and \( \theta_y \) are assumed independent continuous functions, \( U_{ij} \) denotes partial differentiation, and \( \omega, \alpha \) and \( \beta \) are weighting constants of data, derivative constraint and curvature control, respectively. The \( \omega \) constant is commonly used to emphasize the importance of a particular data component. The \( \alpha \) weighting constant stresses the importance of a smooth approximation to the data and implicitly allows for a \( C^1 \)-like construction even with a quadratic base. Finally, the \( \beta \) constant controls the selection of a specific smooth approximation from among the many possible solutions. A finite element–based piecewise representation of \( U(x, y) \) is proposed, domain \( \Omega \) discretized and functional eq (1) restated as

\[
\Phi(U) = \sum_{e=1}^{n_e} \Phi^e(U) = \sum_{e=1}^{n_e} \left( \frac{1}{2N^e_d} \sum_{i=1}^{N^e_d} \omega_i \left[ U^e(x_i, y_i) - u_i \right]^2 \right)
\]

\[
+ \frac{\alpha}{2} \int_{\Omega^e} \left[ (U^e_x - \theta^e_x)^2 + (U^e_y - \theta^e_y)^2 \right] d\Omega
\]

\[
+ \frac{\beta}{2} \int_{\Omega^e} \left[ U^e_{xx}^2 + 2U^e_{xy}^2 + U^e_{yy}^2 \right] d\Omega,
\]

where \( n_e \) is the number of finite elements, \( N^e_d \) is the number of data points falling within an element, and \( U^e, \theta^e_x \) and \( \theta^e_y \) are now functions restricted to an element’s domain. The first and third terms of eq (2), least squared and curvature control, are the more familiar expressions implemented and explored by other investigators using finite element methodologies. The second term of eq (2) is relevant to a specific \( C^1 \)-like construction even with a quadratic base. Finally, the \( \beta \) constant controls the selection of a specific smooth approximation from among the many possible solutions. A finite element–based piecewise representation of \( U^e(x, y) \) is proposed, domain \( \Omega \) discretized and functional eq (1) restated as

\[
U^e = \sum_{i=1}^{3} \xi_i (U^e_i + M_i \theta^e_x + L_i \theta^e_y)
\]

where \( \xi_i \) are the area coordinates as defined by

\[
\xi_i = (a_i + b_i x + c_i y)/(2A)
\]

\[
a_i = x_i y_j - x_j y_i \quad b_i = y_j - y_k \quad c_i = x_k - x_j,
\]

where \( x, y \) are Cartesian coordinates, \( A \) is the area of the triangle and cyclic permutations of \( i, j, k \) are used to evaluate each expression. Components \( M \) and \( L \) are defined as

\[
M_i = 1/2(c_i \xi_j - c_j \xi_i) \quad L_i = 1/2(b_j \xi_i - b_k \xi_j).
\]

The resulting interpolation function eq (3) contains just six independent DoF as discussed in Tessler.\(^1\) Additionally, in eq (2) the value of \( \alpha \) dictates the asymptotic enforcement of \( C^1 \) interelement continuity. The linear functions, \( \theta_x \) and \( \theta_y \), are interpolated from the nodal values using just the area coordinates and become equivalent to the orthogonal gradients everywhere in the element’s domain with a strict enforcement of the second term in eq (2). The more familiar finite element notation for eq (3) is

\[
U^e = N^e u^e,
\]

where \( N^e \) contains the element shape functions and \( u^e \) represents the nodal DoF. A finite element application is formulated as a global set of simultaneous linear equations representing the accumulated effects of local element behavior at the nodes. The response of this particular smoothing element is dictated by three components and summarized by

\[
K^e u^e = (K^e_x + K^e_y + K^e_y) u^e = f^e,
\]

where \( K^e_x, K^e_y \) and \( K^e_y \) symbolically represent the contributions of least squared error, gradient constraint and curvature control, respectively. An expression for each contributor results from the minimization of functional eq (2) with respect to the nodal DoF, \( \partial \Phi = 0 \), and combined with eq (6) leads to the following.

\[
K^e_x = \frac{1}{N^e_d} \sum_{i=1}^{N^e_d} \omega_i N_i^e N_i
\]

\[
K^e_y = \alpha \int_{\Omega^e} \left( N_i^e N_i^e + N_i N_i^e N_i^e N_i^e \right) d\Omega
\]

\[
K^e_y = \beta \int_{\Omega^e} \left( N_i^e N_i^e N_i^e + N_i N_i^e N_i^e N_i^e + N_i N_i^e N_i^e \right) d\Omega
\]

\[
f^e = \frac{1}{N^e_d} \sum_{i=1}^{N^e_d} \omega_i N_i^e u_i,
\]

where

\[
N_i = [\xi_i, M_i, L_i]
\]

The element development continues with an examination of its suitability as a smoothing element candidate. In the next section, the element is applied to data of known distribution and error. The contribution of the gradient constraint term, \( K^e_y \), has been explored in detail in Tessler.\(^1\) The term is thought to instill behavior equivalent to the taut regression spline, and results are shown to be accurate and stable for