Abstract

A concept of finite coverings of continua with a linear order of their members is given. A characterization is obtained of hereditarily locally connected continua which have a finite supremum of cardinalities of the considered coverings.

Finite coverings of continua were introduced in [2]. We recall this concept. Let \( X \) be a metric continuum and \( n \) be a positive integer. By an \( n \)-covering of \( X \) we mean a family \( \mathcal{C} = \{X_1, X_2, \ldots, X_n\} \) of \( n \) its subcontinua such that \( X = \bigcup\{X_i : i \in \{1, \ldots, n\}\} \). An \( n \)-covering of \( X \) is said to be linear if the following conditions hold, where \( i, j \in \{1, \ldots, n\} \): for each \( i \) we have \( X_i \setminus \bigcup \{X_j : j \neq i\} \neq \emptyset \); and \( X_i \cap X_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). In [3] we have defined the linear number \( \lambda(X) \) of a continuum \( X \) as \( \lambda(X) = \sup\{n \in \mathbb{N} : \text{there exists a linear } n\text{-covering of } X\} \). If this number is not finite, we write \( \lambda(X) = \infty \).

In that paper [3] we have presented some sufficient conditions to have infinite linear numbers of locally connected continua. We have obtained a characterization of locally connected continua with finitely many ramification points and with a finite linear number.

Recall (see [1]) that a point \( p \) of a continuum \( X \) is called a ramification point of \( X \) provided that there are at least three arcs in \( X \) each pair of which has \( p \) as the only common point. An arc \( A \) with end points \( p \) and \( q \) which is contained in a space \( X \) is called a free arc provided that \( A \setminus \{p, q\} \) is an open subset of \( X \). We say that a set \( S \) strongly disconnects a space \( X \) provided there are two subsets \( U \) and \( V \) of \( X \) such that \( X \setminus S = U \cup V \) and \( \overline{U} \cap \overline{V} = \emptyset \).

The following proposition and lemma have been shown in [3].

Proposition 1. If a continuum \( X \) contains a free arc that strongly disconnects \( X \), then \( \lambda(X) = \infty \).


Key words and phrases. Arc, chord, continuum, covering, finite, hereditarily, linear, locally connected, ramification point, simple closed curve.
LEMMA 2. Let a locally connected continuum $X$ do not contain any free arc that strongly disconnects $X$, and let $\mathcal{C}_n$ be a linear $n$-covering of $X$. Then for each free arc $pq \subset X$ two members of $\mathcal{C}_n$ are enough to cover $pq$.

To show the main theorem which characterizes hereditarily locally connected continua having finite linear numbers, we need some definitions.

Let a simple closed curve $S$ be given. Each arc $ab$ with $ab \cap S = \{a, b\}$ is called a chord of $S$. We consider a set of $n$ pairwise disjoint chords of $S$ such that the end points of each chord can be labelled as $a_i, b_i$ for $i \in \{1, 2, \ldots, n\}$ in such a way that there are two points $p$ and $q$ of $S$ with the property that one arc, $A$, from $p$ to $q$ in $S$ contains all points $a_i$ with its natural order in the set $A \setminus \{p, q\}$, while the other one, $B$, contains all the points $b_i$ with the same order (from $p$ to $q$) in the set $B \setminus \{p, q\}$. Also we put

\begin{equation}
(3) \quad a_0 = b_1, \quad b_0 = a_1 \quad \text{and} \quad a_{n+1} = b_n, \quad b_{n+1} = a_n
\end{equation}

(this means that the chords $a_1b_1$ and $a_nb_n$ are also considered as $b_0a_0$ and $b_{n+1}a_{n+1}$ respectively).

The above discussed set of $n$ chords is said to be independent provided that there does not exist any chord $ab$ of $S$ such that

\begin{equation}
(4) \quad \text{for some } i, j \in \{0, 1, 2, \ldots, n\} \text{ with } |i - j| > 1 \text{ we have either } a \in a_ia_{i+1} \text{ and } b \in a_ja_{j+1} \cup b_jb_{j+1} \text{ or } a \in b_ib_i+1 \text{ and } b \in b_jb_{j+1}, \text{ and}
\end{equation}

\begin{equation}
(5) \quad \text{for each } i_0 \text{ with } i < i_0 < j \text{ (where } i \text{ and } j \text{ are as in (4) above) we have } a_0 \cap a_{i_0}b_{i_0} = \emptyset.
\end{equation}

LEMMA 6. Let $X$ be a hereditary locally connected continuum which contains a simple closed curve $S$ with an independent set of $k$ chords. Then there exists a linear covering of $X$ with at least $m = \lceil k/2 \rceil + 2$ members (here $\lceil k/2 \rceil$ means the integral part of $k/2$).

PROOF. Consider an independent set of $k$ chords $a_ib_i$, where $i \in \{1, 2, \ldots, k\}$, of $S$. For each $i$ denote by $L_i$ the arc with end points $a_i$ and $b_i$ contained in $S$ and such that for each $j \in \{1, 2, \ldots, k\}$ we have $(a_j, b_j) \cap L_i \neq \emptyset$ if and only if $j \leq i$. Take now two points $c$ and $d$ in $L_1 \setminus \{a_1, b_1\}$ with $c \in a_1d$ and $d \in cb_1$, and let $V_1$ be an open connected set containing the arc $cd$ and such that $V_1 \cap (S \setminus L_1 \cup a_1b_1 \cup a_2b_2) = \emptyset$. Denote by $C_1$ the component of $X \setminus V_1$ containing $S \setminus L_1$, and put $X_1 = X \setminus C_1$. Next take the subarcs $ca_1 \cup a_1a_2 = ca_2$ and $db_1 \cup b_1b_2 = db_2$ of $L_2$ and consider an open connected set $V_2$ in $C_1$ containing the set $(ca_2 \cup db_2 \cup a_1b_1 \cup a_2b_2) \cap C_1$ and such that $V_2 \cap (S \setminus L_3 \cup a_3b_3 \cup a_4b_4) = \emptyset$. Denote by $C_2$ the component of $C_1 \setminus V_2$ which contains $S \setminus L_3$ and define $X_2 = C_1 \setminus C_2$.

In general, for an index $i$ such that $3 \leq i \leq \lceil k/2 \rceil + 1$, take the subarcs $a_{2i-4}a_{2i-2}$ and $b_{2i-4}b_{2i-2}$ of $L_{2i-2}$ and an open connected set $V_i$ in $C_{i-1}$ containing the set

\begin{equation}
(a_{2i-4}a_{2i-2} \cup b_{2i-4}b_{2i-2} \cup a_{2i-3}b_{2i-3} \cup a_{2i-2}b_{2i-2}) \cap C_{i-1}
\end{equation}