CONSTRUCTION OF SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS USING THE FORM OF FORMAL SERIES. III

G. Dosinas and Z. Navickas

8. EXPLICIT CONSTRUCTIONS

8.1. In [1] we defined \((-n)! = \pm \infty\), i.e., \(1/(-n!) = 0\) for \(n \in \mathbb{N}\); the following definition of the ratio of two factorials of non-negative integers will be of further use:

\[
\frac{(-m)!}{(-n)!} \overset{\text{def}}{=} (-1)^{m-n} \frac{(n-1)!}{(m-1)!}.
\]

Such a definition provides a natural extension of operations on factorials of positive integers to factorials of negative integers.

In addition, the coefficient \(\binom{n}{r}\) of the binomial Newton formula can be generalized by

\[
\binom{n}{r} \overset{\text{def}}{=} \frac{n!}{r!(n-r)!}.
\]

for \(n \in \mathbb{Z}, r \in \mathbb{Z}_0\), so that \(\binom{n}{r} = C'_n\) if \(n \geq r \geq 0\), \(n,r \in \mathbb{Z}_0\) (\(C'_n\) denotes the number of combinations of \(n\) elements \(r\) at a time), and \(\binom{n}{r} \overset{\text{def}}{=} 0\) for \(r > n \geq 0\). Finally,

\[
\binom{-n}{r} = \frac{(-1)^{r}(n+r-1)!}{r!(n-1)!}
\]

for \(n \in \mathbb{N}, r \in \mathbb{Z}_0\).

8.2. Define Appell functions \(\sigma^{(k)}_n(x)\) of order \(n \in \mathbb{N}\) by

\[
\sigma^{(k)}_n(x) \overset{\text{def}}{=} \begin{cases} 
0, & k \neq nj, \ j \in \mathbb{Z}, \\
\frac{k!}{j!} x^j, & k = nj, \ j \in \mathbb{Z},
\end{cases}
\]

for arbitrary \(k \in \mathbb{Z}\) and real variable \(x\).

Appell functions \(\sigma^{(k)}_n(x)\) are a generalization of Appell polynomials \(p_k(x), k = 0, 1, 2, \ldots\), determined by a generating equation [3]

\[
\sum_{k \geq 0} \psi_k p_k(x) z^k = A(z) \psi(xg(z)).
\]

---

1The numbering of sections and formulae is continued.


If $A(z) \equiv 1$, $\psi(z) = \exp(z)$, $g(z) = z^n$, $\psi_k = 1/k!$, then the generating equation becomes

$$\sum_{k \geq 0} \frac{1}{k!} \sigma_n^{(k)}(x)z^k = \exp(xz^n), \quad n \in \mathbb{N}.$$ 

By (19) we get that

$$\sigma_n^{(k-n)}(x) = \frac{(-1)^{k-n}(k-1)!}{(k-n)!} \frac{1}{x^k}$$

for $k \in \mathbb{N}$. We define the binomial Newton–Appell formula (generalizing the binomial Newton formula) by the following relations:

$$v_{n,m;r}(x, y) \overset{\text{def}}{=} (\sigma_n^{(r)}(x) + \sigma_m^{(r)}(y))^r \overset{\text{def}}{=} \sum_{j=0}^{\infty} \binom{r}{j} \sigma_n^{(j)}(x)\sigma_m^{(r-j)}(y) \quad (23)$$

or

$$v_{m,n;r}(y, x) \overset{\text{def}}{=} (\sigma_m^{(r)}(y) + \sigma_n^{(r)}(x))^r \overset{\text{def}}{=} \sum_{j=0}^{\infty} \binom{r}{j} \sigma_m^{(j)}(y)\sigma_n^{(r-j)}(x) \quad (24)$$

for $r \in \mathbb{Z}$. From the definitions (23) and (24) we directly get the following properties of the binomial Newton–Appell formula:

**Property 1.** By the definition of the function $u_r(x, y; \alpha, \beta)$ (cf. [1], formula (3)) the following commutative identity holds:

$$v_{n,m;r}(\alpha x, \beta y) = v_{m,n;r}(\beta y, \alpha x) = r!u_r(x, y; \alpha, \beta) \quad (25)$$

for $r \in \mathbb{Z}$.

≤ Setting $j = nk$, $r - j = ml$, we have

$$\left(\sigma_n^{(r)}(\alpha x) + \sigma_m^{(r)}(\beta y)\right)^r = \sum_{j=0}^{\infty} \binom{r}{j} \sigma_n^{(j)}(\alpha x)\sigma_m^{(r-j)}(\beta y)$$

$$= \sum_{k \in \mathbb{Z}_0} \frac{r!}{(kn)!(r-kn)!} \frac{(kn)!}{k!} \sigma_n^{(k-n)}(\alpha x)^{k} \sigma_m^{(r-k)}(\beta y)$$

$$= r! \sum_{nk+ml=r} \frac{(\alpha x)^k}{k!} \frac{(\beta y)^l}{l!}$$

$$= r!u_r(x, y; \alpha, \beta); \quad j, l \in \mathbb{Z}_0.$$ 

Now checking the commutativity of Appell functions for $r \in \mathbb{Z}_0$ is easy.

**Remark 1.** $v_{1,1;r}(\alpha x, \beta y) = (\alpha x + \beta y)^r$, $r \in \mathbb{Z}$; however, for $n, m \neq 1$ and $r_1, r_2 \in \mathbb{Z}$, we have

$$\left(\sigma_n^{(r_1)}(\alpha x) + \sigma_m^{(r_2)}(\beta y)\right)^{r_1+r_2} \neq \left(\sigma_n^{(r_1)}(\alpha x) + \sigma_m^{(r_2)}(\beta y)\right)^{r_1} \left(\sigma_n^{(r_2)}(\alpha x) + \sigma_m^{(r_1)}(\beta y)\right)^{r_2}.$$ 

**Property 2.** Denote

$$u_{-r}^{(r)}(x, y; \alpha, \beta) \overset{\text{def}}{=} \sum_{n \in \mathbb{N}, m \in \mathbb{N}; j \in \mathbb{Z}_0, l \in \mathbb{N}} \frac{(\alpha x)^j (-1)^j (l - 1)!}{j! (\beta y)^l} \quad (26)$$

and

$$u_{-r}^{(-r)}(x, y; \alpha, \beta) \overset{\text{def}}{=} \sum_{n \in \mathbb{N}, m \in \mathbb{N}; j \in \mathbb{Z}_0, l \in \mathbb{N}} \frac{(-1)^l (l - 1)! (\beta y)^j}{(\alpha x)^j} \frac{1}{j!} \quad (27)$$