ON THE ASYMPTOTICS OF ONE-SIDED LARGE DEVIATION PROBABILITIES

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Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed real random variables with a common distribution function (d.f.) $F(t), t \in \mathbb{R}$. Let

$$S_n = X_1 + \cdots + X_n, \quad n = 1, 2, \ldots.$$ 

The following elegant one-sided large deviation theorem is due to S. V. Nagaev [1]:

If $M X_1 = 0$ and

$$1 - F(t) \sim t^{-\alpha} h(t) \quad \text{as} \quad t \to \infty,$$

where $\alpha > 1$ and $h(t)$ is a slowly varying function, then

$$P(S_n \geq t) \sim n(1 - F(t))$$

as $n \to \infty$ and $\liminf_{n \to \infty} n/t < \infty$ (here $a_n \sim b_n$ means $\lim_{n \to \infty} a_n/b_n = 1$).

In the present paper, we generalize this result, namely, we prove that relation (1) is still true, if the d.f. $F$ belongs to the class $\mathcal{D}$ of dominated-variation distributions.

**Definition 1.** The distribution function $F$ belongs to the class $\mathcal{D}$ of dominated-variation distributions if its tail $F(t) = 1 - F(t)$ satisfies

$$\limsup_{t \to \infty} F(t)/F(2t) < \infty. \quad (2)$$

When dealing with our problem, it appears to be natural to confine ourselves to the space $\mathcal{L}$.

**Definition 2.** The d.f. $F$ is said to belong to the class $\mathcal{L}$ if

$$\lim_{t \to \infty} \frac{F(t - y)}{F(t)} = 1, \quad \forall y \in \mathbb{R}. \quad (3)$$

It follows from these definitions that the class of distributions with regularly varying right tails, i.e., the class investigated in [1], is contained in $\mathcal{D} \cap \mathcal{L}$.

Define the hazard function

$$R(t) = -\log(1 - F(t)).$$

Assume that there exists a non-negative function $q: \mathbb{R}^+ \to \mathbb{R}$ such that

$$R(t) = R(0) + \int_0^t q(u) \, du, \quad t \in \mathbb{R}^+.$$
The function \( q \) is called the hazard rate of \( F(t), \ t \in \mathbb{R}^+ \).
It is well known (see [2]) that if for some \( F \in \mathbb{L} \) the hazard rate \( q \) or \( \lim_{t \to \infty} q(t) \) does not exist, one can always construct a distribution function \( F_0 \) such that \( F_0(t) \sim F(t) \) as \( t \to \infty \) and \( q_0(t) \to 0 \) as \( t \to \infty \).
It is also known that if \( \limsup_{t \to \infty} qt(t) < \infty \), then \( F(t) \in \mathbb{D} \cap \mathbb{L} \) ([2, Theorem 3.3]).
On the other hand, if the hazard rate \( q \) is nonincreasing, then the statements
\[
F(t) \in \mathbb{D} \cap \mathbb{L}
\]
and
\[
\limsup_{t \to \infty} qt(t) < \infty
\]
are equivalent (see [2, Cor. 3.4]).
It is well known (see [3]) that if \( F(t) \in \mathbb{D} \), then for each \( \alpha' > \alpha(F), \ \beta' < \beta(F) \) and each \( c (1 \leq c \leq \infty) \) there exist constants \( A, B \) and \( x_0 \) such that
\[
A t^\beta' \leq \bar{F}(tx)/\bar{F}(x) \leq B t^\alpha', \quad 1 \leq t \leq c, \quad x > x_0
\]
(here \( \alpha(F) \) and \( \beta(F) \) are, respectively, upper- and lower-Matuszewska indices).
Define
\[
\alpha = \liminf_{t \to \infty} R(t)/\log t, \quad \beta = \liminf_{t \to \infty} (- \log F(t))/\log |t|.
\]

The following theorem presents the main result of this paper.

**THEOREM.** Assume that
1) \( MX_1 = 0 \);
2) \( \limsup_{t \to \infty} qt(t) < \infty \);
3) \( \alpha > 1, \ \beta > 1 \).

Then
\[
P(S_n \geq t) \sim n(1 - F(t))
\]
as \( n \to \infty \) and \( \limsup_{n \to \infty} n/t < \infty \).

**Proof.** Put \( y = t/R(t) \). Let \( \xi \) be the number of summands \( X_k, \ k = 1, \ldots, n \), in \( S_n \) such that \( X_k \geq y \). Since the random variable \( \xi \) has the Bernoulli distribution with parameters \( n \) and \( F(y) \), we may write
\[
P(S_n \geq t) = \sum_{k=0}^n P(S_n \geq t, \ \xi = k).
\]
Letting
\[
Z_k = \begin{cases} X_k, & \text{for } X_k < y, \\ 0, & \text{for } X_k \geq y, \end{cases}
\]
\[
T_n = \sum_{k=1}^n Z_k,
\]
we have for any \( s > 0 \) and any \( u > 0 \)
\[
P(S_n \geq u, \ \xi = 0) = P(S_n \geq u, \ \max_{k \leq n} X_k < y) \leq P(T_n \geq u)
\]
\[
= P(e^{T_n} \geq e^{su}) \leq e^{-su} \mathbb{E}e^{su}
\]
\[
= e^{-su} \mathbb{E} \left( \prod_{k=1}^n e^{Z_k} \right) = e^{-s} n(\mathbb{E}e^{Z_1})^n.
\]

(4)