AXIOMATIZATION OF CERTAIN PROBLEMS OF MINIMIZATION

SUMMARY. In Part I of this paper, an abstract analogue of the minimization problem for Boolean functions and of the notion of prime implicant is defined, so that this general problem can be solved in the same steps as in the classical case: 1) determination of the prime implicants; 2) determination of all the solutions made up of prime implicants. In Part II it is shown that the classical minimization problem, as well as certain set-theoretical and graph-theoretical problems are particular cases of the general problem defined in Part I.

The problem of simplifying the disjunctive normal form of a Boolean function is one of the basic problems in switching algebra. As it is well-known (see, for instance, [2]), there are many algorithms which, developing W. V. Quine’s idea [21], [22] carry out the simplification in two steps:

1°) determination of the so-called prime implicants,
2°) determination of a solution made up of prime implicants.

Prof. Gr. C. Moisil [15], [16] has used an axiomatic approach in order to show that the above method applies not only to functions written as a disjunction of conjunctions (or conjunction of disjunctions), but is also applicable when the given Boolean function is expressed by other interpolation formulas, which utilise the n-ary Sheffer functions and correspond to circuits with electronic devices; the same procedure applies to two-valued functions with three-valued or five-valued arguments, describing the actual operation of a switching circuit [17], [18], [19], [9], [14].

On the other hand, I. B. Pyne and E. J. McCluskey, T. L. Maistrove, P. L. Ivănescu, I. Rosenberg, S. Median and the author have shown [20], [12], [6], [9], [14] that the above problems of minimization reduce to a certain particular case of a problem of pseudo-Boolean programming (this means the problem of finding the minimum of a certain real — (or integer — ) valued function whose variables are bivalent and must satisfy certain constraints expressed by Boolean equations [5]). It was also shown [6], [4], [23] that a certain set-theoretical problem, as well as the problem of finding the chromatic number of a graph, reduce to the same problem of pseudo-Boolean programming.

The above remarks were the starting point of the present work which has a double purpose:
I) to define an abstract analogue on the minimization problem and of the notion of prime implicant so that, under hypotheses as general as possible, this problem may be solved in the two steps 1°, 2°) indicated above.¹

II) to establish a deeper relationship among all the particular problems mentioned above, by showing that they are particular cases of the general problem defined at I).

According to purposes I) and II), this paper is divided into two parts.

Part I consists of three sections. In § 1, the abstract analogues of the minimization problem and of the notion of prime implicants are defined in terms of the theory of quasi-ordered sets. Theorems 2 and 3 establish conditions under which this abstract problem has a solution made up of prime implicants. Theorems 4, 5 and Corollary 1 from § 2 establish conditions under which the prime implicants defined in § 1 may be found by using an analogue of the Quine-McCluskey algorithm for the prime implicants of a Boolean function [21], [22], [13]. Finally, Corollary 2 and Theorem 6 from § 3 establish conditions under which, supposing the prime implicants already known, the solutions made up of prime implicants (which exist, according to § 1), may be found by means of pseudo-Boolean programming [5].

Part II is a survey of certain important particular cases of the problem studied in Part I.

Prof. Gr. C. Moisil [15], [16] has found eight interpolation formulas for Boolean functions (using conjunction, disjunction or Sheffer functions) so that, starting from each of them, we can apply the Quine-McCluskey procedure for finding the simplest forms without parentheses, of the given Boolean functions. In other words, we are faced with eight minimization problems, say $M_1, M_2, \ldots, M_8$, so that each of them can be solved with the Quine procedure. In [16] an axiomatic model for these eight problems was given, in terms of equality algebras satisfying certain postulates. Roughly speaking, in such an algebra Boolean expressions are replaced by purely formal expressions and Boolean functions by equivalence classes of these expressions with respect to a certain congruence relation. If $\omega$ and $\theta$ are the signs which take the place of conjunction and disjunction respectively, then the analogues of the elementary conjunctions are expressions like $\omega(x_{i_1}^{x_{i_1}} \ldots x_{i_r}^{x_{i_r}})$, with $x_{i_1}, \ldots, x_{i_r} = 0$ or 1, while the analogues of disjunctive normal forms are expressions like $a_1 \theta \ldots \theta a_s$, where each $a_i$ is of the form $x_{i_1}^{x_{i_1}} \ldots x_{i_r}^{x_{i_r}}$.² Denote by $E$ the minimization problem for these equality algebras. In the same paper [16], it is shown that the Quine procedure can also be applied in the case of 2-valued or $n$-valued Quine algebras, which are more general than the above defined equality algebras; denote the corresponding minimization problems by $2Q$ and $nQ$, respectively.

In the papers [6] and [23], a certain set-theoretical problem, say $S$, was defined, related to the minimization problem and including the problem of finding the chro-

¹ Of course, we have no claim that our answer to this problem I) is the single possible solution.
² In § 5, we shall assume the reader is familiar with [15] or [16].