WEYL GROUPS AS GALOIS GROUPS OF A REGULAR EXTENSION OF THE FIELD Q

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For a finite group $G$, $\text{Gal}_T(G)$ denotes the property that there exists a regular Galois extension of the rational function field $\mathbb{Q}(T)$ over the field of rationals $\mathbb{Q}$, with a Galois group $G$. This property is established to be satisfied by all Weyl groups except the type $F_4$, from which it follows that it holds also for Chevalley groups $C_3(2)$ and $D_4(2)$.

Denote by $\text{Gal}_T(G)$ the following property of a finite group $G$: there exists a regular Galois extension of the rational function field $\mathbb{Q}(T)$ over the field of rationals $\mathbb{Q}$, with a Galois group $G$. This property implies that there exist infinitely many Galois extensions of $\mathbb{Q}$ with a Galois group $G$ which are linearly separated from each other (see [1]). In this article we prove the following:

THEOREM. If $W$ is a Weyl group of type not $F_4$, then the property $\text{Gal}_T(W)$ is satisfied.

In proving the theorem, we rely on the results of Belyi [2]. Namely, for all Weyl groups (except the types $D_{2k}$ and $F_4$), we use a uniform method to find a rigid (rational) triplet, and then our theorem follows directly from [2].

The fact that the theorem is valid for the case where a Weyl group $W$ of type $A_i$ is isomorphic to the symmetric group $S_{i+1}$ was known as early as the time of Hilbert [3] (see also the survey in [1], where an example of a rigid rational triplet for $S_{i+1}$ is given).

Since the property $\text{Gal}_T(W)$ is satisfied for Weyl groups of types $E_7$ and $E_8$, we can infer the following:

COROLLARY. If $G$ is a Chevalley group of type $C_3(2)$ or $D_4(2)$, then the property $\text{Gal}_T(G)$ is satisfied.

1. Throughout, $\Phi$ is a reduced indecomposable root system, $\pi = \{r_1, r_2, \ldots, r_l\}$ is the set of fundamental roots of $\Phi$ such that $r_1$ is a short root, and $W$ is the Weyl group of type $\Phi$ generated by fundamental reflections $w_{r_i}$. The alignment of fundamental roots is shown in the following Dynkin’s diagram:

We distinguish \( c = w_{r_1} w_{r_2} \ldots w_{r_l} \), one of the Coxeter elements.

**Proposition 1** (see [4, 5]). Suppose that \( W \) is of type distinct from \( B_1, D_{2k}, \) or \( F_4 \). Then \( W = \langle c, w_{r_2} \rangle \).

**Proposition 2** (see [6]). Let

\[
s_i = w_{r_1} w_{r_{i-1}} \ldots w_{r_{i+1}}(r_i), \quad i = 1, \ldots, l,
\]

and let \( \Phi_i \) be the orbit of the root \( s_i \) with respect to the group \( \langle c \rangle \). Then the subsets \( \Phi_i \) intersect pairwise trivially and exhaust all the orbits of \( \langle c \rangle \) in \( \Phi \).

2. Let \( C_1, \ldots, C_k \) \((k \geq 3)\) be conjugacy classes of a finite group \( G \). Denote by \( P = P(C_1, \ldots, C_k) \) the set of \( k \)-tuples \((g_1, \ldots, g_k) \in C_1 \times \ldots \times C_k\) such that \( g_1 g_2 \ldots g_k = 1 \) and \( G = \langle g_1, \ldots, g_k \rangle \). The collection \( (C_1, \ldots, C_k) \) is called rigid if \( P \) is not empty and \( G \) acts transitively (by conjugation) on \( P \).

A conjugacy class \( C \) of a group \( G \) is said to be rational if each character of \( G \) takes a rational value on \( C \).

**Proposition 3** (see [2]). Suppose that a finite group \( G \) satisfies the following conditions:

1. \( G \) has a rigid rational triplet \((C_a, C_b, C_{a+b-1})\);

2. \( Z(G) \) is a direct summand in \( N_G((a)) \).

Then the property \( \text{Gal}_P(G) \) is satisfied.

Here \( C_x \) is a conjugacy class with a representative \( x \), \( N_G(A) \) is the normalizer of a subgroup \( A \) in a group \( G \), and \( Z(G) \) is the center of \( G \).

3. Proof of the theorem. Suppose that the type of \( W \) is distinct from \( B_1, D_{2k}, \) or \( F_4 \) and let \( C_1, C_2, \) and \( C_3 \) be the conjugacy classes of \( W \) with representatives \( w_{r_1}, c, \) and \( c^{-1}w_{r_1} \), respectively. It is well known that all conjugacy classes of a Weyl group are rational (see, e.g., [1]). Therefore, by virtue of Proposition 3, the theorem follows from the following two lemmas.

**Lemma 1.** The triple \((C_1, C_2, C_3)\) is a rigid triplet of the group \( W \).

**Proof.** By Proposition 1 we have \( W = \langle w_{r_1}, c \rangle \). Let \( a_i \in C_i \) and \( W = \langle a_1, a_2, a_3 \rangle \). Without loss of generality, we may assume that \( a_2 = c \) and \( a_1 = w_r \) for some root \( r \in \Phi \). It is sufficient to show that \( c^l(r) = r_1 \) for some \( i \in Z \).

By Proposition 2, as representatives of the orbits of the group \( \langle c \rangle \) in \( \Phi \) we can take the following elements:

- \( r_1 + \ldots + r_1, r_2 + \ldots + r_1, \ldots, r_1 \) for \( \Phi \neq D_1, E_6 \);
- \( r_1 + r_3 + \ldots + r_1, r_2 + r_3 + \ldots + r_1, \ldots, r_1 \) for \( \Phi = D_7 \);
- \( r_1 + \ldots + r_1, r_1-3 + \ldots + r_1, r_1-2 + \ldots + r_1, r_1, r_1 \) for \( \Phi = E_6 \).

So we can assume that \( r \) coincides with one of the roots indicated above.

Suppose that \( r \) is one of the following:

- \( r_1 + \ldots + r_1 \) or \( r_1 \) for \( \Phi = A_1 \);