WEYL GROUPS AS GALOIS GROUPS OF A REGULAR
EXTENSION OF THE FIELD Q

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For a finite group \( G \), \( \text{Gal}_T(G) \) denotes the property that there exists a regular Galois extension of the rational function field \( \mathbb{Q}(T) \) over the field of rationals \( \mathbb{Q} \), with a Galois group \( G \). This property is established to be satisfied by all Weyl groups except the type \( F_4 \), from which it follows that it holds also for Chevalley groups \( C_3(2) \) and \( D_4(2) \).

Denote by \( \text{Gal}_T(G) \) the following property of a finite group \( G \): there exists a regular Galois extension of the rational function field \( \mathbb{Q}(T) \) over the field of rationals \( \mathbb{Q} \), with a Galois group \( G \). This property implies that there exist infinitely many Galois extensions of \( \mathbb{Q} \) with a Galois group \( G \) which are linearly separated from each other (see [1]). In this article we prove the following:

THEOREM. If \( W \) is a Weyl group of type not \( F_4 \), then the property \( \text{Gal}_T(W) \) is satisfied.

In proving the theorem, we rely on the results of Belyi [2]. Namely, for all Weyl groups (except the types \( D_{2k} \) and \( F_4 \)), we use a uniform method to find a rigid (rational) triplet, and then our theorem follows directly from [2].

The fact that the theorem is valid for the case where a Weyl group \( W \) of type \( A_n \) is isomorphic to the symmetric group \( S_{n+1} \) was known as early as the time of Hilbert [3] (see also the survey in [1], where an example of a rigid rational triplet for \( S_{n+1} \) is given).

Since the property \( \text{Gal}_T(W) \) is satisfied for Weyl groups of types \( E_7 \) and \( E_8 \), we can infer the following:

COROLLARY. If \( G \) is a Chevalley group of type \( C_3(2) \) or \( D_4(2) \), then the property \( \text{Gal}_T(G) \) is satisfied.

1. Throughout, \( \Phi \) is a reduced indecomposable root system, \( \pi = \{r_1, r_2, \ldots, r_l\} \) is the set of fundamental roots of \( \Phi \) such that \( r_1 \) is a short root, and \( W \) is the Weyl group of type \( \Phi \) generated by fundamental reflections \( w_{r_i} \). The alignment of fundamental roots is shown in the following Dynkin's diagram:

We distinguish $c = w_{r_1}w_{r_2} \ldots w_{r_l}$, one of the Coxeter elements.

Proposition 1 (see [4, 5]). Suppose that $W$ is of type distinct from $B_l, D_{2k}$, or $F_4$. Then $W = \langle c, w_{r_1} \rangle$.

Proposition 2 (see [6]). Let

$$s_i = w_{r_1}w_{r_{i-1}} \ldots w_{r_{i+1}}(r_i), \quad i = 1, \ldots, l,$$

and let $\Phi_i$ be the orbit of the root $s_i$ with respect to the group $\langle c \rangle$. Then the subsets $\Phi_i$ intersect pairwise trivially and exhaust all the orbits of $\langle c \rangle$ in $\Phi$.

2. Let $C_1, \ldots, C_k$ $(k \geq 3)$ be conjugacy classes of a finite group $G$. Denote by $P = P(C_1, \ldots, C_k)$ the set of $k$-tuples $(g_1, \ldots, g_k) \in C_1 \times \ldots \times C_k$ such that $g_1g_2 \ldots g_k = 1$ and $G = \langle g_1, \ldots, g_k \rangle$. The collection $(C_1, \ldots, C_k)$ is called rigid if $P$ is not empty and $G$ acts transitively (by conjugation) on $P$.

A conjugacy class $C$ of a group $G$ is said to be rational if each character of $G$ takes a rational value on $C$.

Proposition 3 (see [2]). Suppose that a finite group $G$ satisfies the following conditions:

1. $G$ has a rigid rational triplet

$$\langle C_a, C_b, C_{b-1} \rangle;$$

2. $Z(G)$ is a direct summand in $N_G((a))$. Then the property $G_{\text{Gal}}(G)$ is satisfied.

Here $C_x$ is a conjugacy class with a representative $x$, $N_G(A)$ is the normalizer of a subgroup $A$ in a group $G$, and $Z(G)$ is the center of $G$.

3. Proof of the theorem. Suppose that the type of $W$ is distinct from $B_l, D_{2k}$, or $F_4$ and let $C_1, C_2, C_3$ be the conjugacy classes of $W$ with representatives $w_{r_1}, c$, and $c^{-1}w_{r_1}$, respectively. It is well known that all conjugacy classes of a Weyl group are rational (see, e.g., [1]). Therefore, by virtue of Proposition 3, the theorem follows from the following two lemmas.

**Lemma 1.** The triple $(C_1, C_2, C_3)$ is a rigid triplet of the group $W$.

Proof. By Proposition 1 we have $W = \langle w_{r_1}, c \rangle$. Let $a_i \in C_i$ and $W = \langle a_1, a_2, a_3 \rangle$. Without loss of generality, we may assume that $a_2 = c$ and $a_1 = w_r$ for some root $r \in \Phi$. It is sufficient to show that $c^i(r) = r_1$ for some $i \in Z$.

By Proposition 2, as representatives of the orbits of the group $\langle c \rangle$ in $\Phi$ we can take the following elements:

- $r_1 + \ldots + r_1, r_2 + \ldots + r_1, r_3 + \ldots + r_1$ for $\Phi \neq D_l, E_l$;
- $r_1 + r_3 + \ldots + r_1, r_2 + r_3 + \ldots + r_1, r_3 + \ldots + r_1$ for $\Phi = D_l$;
- $r_1 + \ldots + r_1, r_1 + \ldots + r_1, r_1 + \ldots + r_1, \ldots + r_1$ for $\Phi = E_l$.

So we can assume that $r$ coincides with one of the roots indicated above.

Suppose that $r$ is one of the following:

- $r_1 + \ldots + r_1$, or $r_1$ for $\Phi = A_l$;