Some Inequalities for Starlike Functions

By

St. Ruscheweyh, Würzburg, and V. Singh, Patiala, India

(Received 13 February 1984)

Abstract. For normalized starlike univalent functions in the unit disc we derive upper and lower estimates for certain differential expressions of $n$-th order (including $|zf'(z)/f(z)|$ for $n = 1$) in terms of $|f(z)|$. Our results generalize and/or improve earlier ones by Twomey and Singh. The operators and methods applied come from the theory of the Peschl—Bauer differential equation.

1. Introduction

Let $D$ be the unit disc $\{z:|z|<1\}$ and $S^*$ the family of normalized starlike univalent functions in $D$, i.e.$f \in S^*$ iff $f$ analytic in $D$, $f(0)=0$, $f'(0)=1$, and $\Re[zf'(z)/f(z)]>0$ in $D$. The aim of this note is to obtain new inequalities for $g(z)=zf'(z)/f(z)$ and certain combinations of its derivatives up to the $n$-th order in terms of $|f(z)|$. We mention two previous results.

Theorem A. (TWOMEY [3]) For $f \in S^*$ and $z \in D$ we have ($r=|z|$)

$$
\left|\frac{zf'(z)}{f(z)}\right| \leq 1 + \frac{r \log[(1+r)^2|f(z)|/r]}{(1-r)\log\frac{1+r}{1-r}}.
$$

Theorem B. (V. SINGH [2]) Under the same assumptions we have

$$
\frac{1-r^2}{r}|f(z)| \leq \Re\frac{zf'(z)}{f(z)} \leq \frac{1-r}{1+r} + \frac{2r \log[(1+r)^2|f(z)|/r]}{(1-r^2)\log\frac{1+r}{1-r}}.
$$

We shall give a new simple proof for the right hand side of (2) and show that (2) contains an improvement of (1), at least for small $r$. Furthermore, our proof extends to certain differential operators which are related to the class of $n$-harmonic functions. These are the solutions of the so-called Peschl—Bauer equation

$$
(1-|z|^2)^2 w_{zz} - n(n+1)w = 0, \quad n \in \mathbb{N}.
$$
For a complete account of that theory see K.W. Bauer and St. Ruscheweyh [1]. We shall need only the following results:

**Theorem C.** \( w \) is a real \( n \)-harmonic function in \( D \) iff there exists an analytic function \( g \) in \( D \) such that

\[
\frac{w}{\Re S_n g} = \sum_{k=0}^{n} \binom{n+k}{n} \frac{r^2}{1-r^2} \frac{(z^n g)^{(n-k)}}{z^k (n-k)!}.
\]

**Theorem D.** \( w \) is a positive \( n \)-harmonic function in \( D \) with \( w(0) = 1 \) iff (4) holds with \( \Re g(z) > 0 \) in \( D \), \( g(0) = 1 \). If \( g \) has the Herglotz representation

\[
g(z) = \frac{2\pi}{\beta} \frac{1 + ze^{it}}{1 - ze^{it}} d\mu(t)
\]

with a certain probability measure \( \mu \) on \([0, 2\pi]\) then

\[
w(z) = \int_0^{2\pi} \frac{(1 - r^2)^{n+1}}{|1 - ze^{it}|^2} d\mu(t).
\]

In particular,

\[
\left( \frac{1 - r}{1 + r} \right)^{n+1} \leq w(z) \leq \left( \frac{1 + r}{1 - r} \right)^{n+1}, \quad z \in D,
\]

for any such solution.

Although the following theorem deals with starlike functions it is clear that the estimate (8) can as well be considered as an improvement over (7) for positive \( n \)-harmonic functions. We use the abbreviation \( A = \left(\frac{1+r}{1-r}\right)^2 \).

**Theorem 1:** Let \( f \in S^* \). Then for \( z \in D \), \( r = |z| \), we have

\[
L_n \leq \Re \left( S_n \frac{zf''}{f'} \right)(z) \leq U_n,
\]

where

\[
L_n = A^{-\frac{n+1}{2}} \left\{ 1 - \frac{A^{n+1} - 1}{\log A} \log \left( \frac{(1-r)^2}{r} |f(z)| \right) \right\}^{-1},
\]

\[
U_n = A^{\frac{n+1}{2}} \left\{ 1 + \frac{A^{n+1} - 1}{\log A} \log \left( \frac{(1+r)^2}{r} |f(z)| \right) \right\}.
\]