Counterexamples to a Conjecture of Mader about Cycles Through Specified Vertices in $n$ - Edge - Connected Graphs

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Abstract. For each odd $n \geq 3$, we construct $n$ - edge - connected graphs $G$ with the following property: There are two vertices $u$ and $v$ in $G$ such that for every cycle $C$ in $G$ passing through $u$ and $v$ the graph $G - E(C)$ is not $(n - 2)$ - edge - connected. Here $E(C)$ denotes the set of edges of $C$, and a cycle is allowed to pass through a vertex more than once.

We consider graphs which are finite, undirected, without loops and in which multiple edges are possible. Let $G = (V, E)$ be a graph. $V(G) = V$ and $E(G) = E$ denote the set of the vertices of $G$ and the set of the edges of $G$ respectively. If $X$, $Y \subseteq V$ and $X \cap Y = \emptyset$, then let $[X, Y]_G$ be the set of all edges of $G$ connecting a vertex of $X$ with a vertex of $Y$. Moreover let $\delta(G; X, Y) := |[X, Y]_G|$ and $\delta(G; X) := \delta(G; X, V - X)$. When using these notations, we also write $x$ and $y$ instead of $X$ and $Y$ respectively, if $X = \{x\}$ and $Y = \{y\}$ respectively.

Paths and cycles in $G$ are allowed to pass through a vertex more than once, but using an edge more than once is forbidden. $G$ is called $n$ - edge - connected, if for each distinct $x, y \in V$, there are at least $n$ edge - disjoint paths in $G$ connecting $x$ and $y$.

Mader [1] conjectured that the following statement ($\ast$) is true for each $n \geq 4$:

($\ast$) Let $G$ be an $n$ - edge - connected graph and $u, v$ be vertices of $G$. Then there exists a cycle $C$ in $G$ passing through $u$ and $v$ such that $G - E(C)$ is $(n - 2)$ - edge - connected.

This statement was proved by Mader [1] for $n = 4$ and by Okamura [3] for each even $n \geq 6$. In this paper for each odd $n \geq 3$, we give counterexamples to ($\ast$).

Theorem 1. If $n \geq 3$ is odd, then there exists an $n$ - edge - connected graph $G$ of order $2(n + 3)/2$, which contains two vertices $u$ and $v$ of distance three, such that for each cycle $C$ passing through $u$ and $v$, $G - E(C)$ is not $(n - 2)$ - edge - connected.

In Theorem 1 the distance between $u$ and $v$ is the minimum possible, because Okamura [2] and Mader [1] proved the following: If $G$ is $n$ - edge - connected and
Let \( G = (V, E) \) be a graph. If \( X \subseteq V \) and if \( x \in X \), we let \( G^* = G/X \to x \) be the graph obtained from \( G \) by contracting \( X \) to \( x \). We do this in such a way, that \( V(G^*) = (V - X) \cup \{x\} \) and \([X, y]_G^* = [x, y]_G\) for each \( y \in V - X \). If \( X_1, \ldots, X_k \subseteq V \) are pairwise disjoint and if \( x_i \in X_i \) for each \( i \in \{1, \ldots, k\} \), we define \( G/X_1 \to x_1, \ldots, X_k \to x_k \) inductively by \( G/X_1 \to x_1 \to x_2 \to \cdots \to x_k \).

The following lemma is well known and easily seen to hold:

**Lemma 1.** Let \( G \) be a graph, \( X \subseteq V(G), x \in X, y \in V - X \) and \( n := \delta(G; X, V - X) \). Moreover let \( G/X \to x \) and \( G/(V - X) \to y \) be \( n \)–edge–connected. Then \( G \) is also \( n \)–edge–connected.

In the following let \( n \geq 3 \) be a fixed odd number and \( \alpha := \frac{n - 1}{2} \). Moreover let \( u \) and \( v \) be two distinct fixed vertices. We call a graph \( G \) admissible, if \( u, v \in V(G) \) and \( \delta(G; x) = n \) for each \( x \in V(G) \) and if \( G \) is \( n \)–edge–connected. A cycle \( C \) in \( G \) is called nice, if \( u, v \in V(C) \) and if \( G - E(C) \) is \((n - 2)\)–edge–connected.

In figures of graphs \( x \leftrightarrow y \) always indicates exactly \( m \) edges connecting \( x \) and \( y \). For each \( l \in \{1, 2, \ldots, \alpha\} \), let \( H_l \) be a graph given by the figure below, i.e. \( V(H_l) = \{u, v, a, b\} \), \( \delta(H_l; u, v) = \delta(H_l; a, b) = \alpha, \delta(H_l; u, a) = \delta(H_l; v, a) = l \) and \( \delta(H_l; u, a) = \delta(H_l; v, b) = \alpha + 1 - l \).

The following lemma is easily seen to hold.