AN ASYMPTOTIC OF HIGH ORDER MOMENTS OF RENEWAL

A. Baltrūnas

1. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed positive random variables, whose distribution $F$ has an absolutely continuous component. Let $S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$, $n > 0$, be partial sums. One of the central objects in renewal theory is the family

$$N(t) = \inf\{n \geq 1: S_n > t\}, \quad t \geq 0,$$

of first passage times for $S_n$, $n \geq 0$. The expectation $H(t) = MN(t)$, $t \geq 0$, is the so-called renewal function.

A number of authors have investigated the asymptotic behavior of $H(t)$ as $t \to \infty$. As was pointed out by many authors (see [1] and references therein), instead of the investigation of high order moments of renewal one can study the so-called $\Phi$-moments of renewal

$$\Phi_n(t) = M\{N(t)(N(t) + 1) \cdots (N(t) + n - 1)\}, \quad n = 1, 2, \ldots,$$

because the Laplace transform for the function $\Phi_n(t)$ has a simple form

$$\Phi_n(s) = \int_0^\infty e^{-st} \Phi_n(t) \, dt = n!/(s(1 - \hat{f}(s))^n),$$

where $\hat{f}(s) = M\exp(-sX_1)$, $s \geq 0$. For $n = 1$, we have $\Phi_1 \equiv H$.

The asymptotic of $\Phi_n$-moments of renewal, when there exists a moment of the order $n + 1$ of the random variable $X_1$, has been exhaustively investigated in [1, 2]. In the present paper we investigate the asymptotic high order of moments of renewal for a large class of distributions, without the request for the existence of the $n + 1$ moment of the random variable $X_1$.

2. Let $\mathcal{F}$ be a set of functions, integrable on $[0, \infty)$, and let $C$ be a set of functions, differentiable on $[0, \infty)$, except for a countable set of points. For each $g \in \mathcal{F}$ let us ascribe its corresponding Laplace transform

$$\hat{g}(s) = \int_0^\infty e^{-sx} g(x) \, dx, \quad s > 0.$$

For any two functions $h$ and $f$ of $\mathcal{F}$ we define their convolution $h \ast f$ by

$$h \ast f(x) = \int_0^x h(x - u)f(u) \, du, \quad x \in [0, \infty).$$

The $m$-fold convolution of $f \in \mathcal{F}$ with itself will be denoted by $f^{*m}$, $m \in \mathbb{N}$.
Two functions $h$ and $g$ are said to be equivalent ($h \sim g$) if $\lim_{t \to \infty} h(t)/g(t) = 1$. We call two functions $h$ and $g$ weakly equivalent, denoted by $h \ll g$, if there exist $m, M \in (0, \infty)$ such that $m \leq h(t)/g(t) \leq M$ for all $t \in (0, \infty)$.

Denote

$$C_0 = \{f: f \in C, f(0) = 0\}.$$ 

Let us define the operator $\Delta: \mathcal{F} \to C_0$ by the formula

$$\Delta f(x) = \int_0^x f(u) \, du, \quad x \geq 0.$$ 

The operator $\Delta^{-1}$ is the inverse operator of $\Delta$.

Denote

$$R_0(t) = P(X_1 > t),$$

$$R_m(t) = \int R_{m-1}(u) \, du, \quad m = 1, 2, \ldots$$

(if $R_m(0) < \infty$).

3. Define

$$U(t) = \int_0^t R_0(u) \, du, \quad t \geq 0.$$ 

Assume that the function $U(t)$, $t \geq 0$, varies slowly as $t \to \infty$.

Then we have the following result.

**THEOREM 1.** If $U(t)$ varies slowly as $t \to \infty$, then for every positive integer $n$

$$\Phi_n(t) \sim t^n / U^n(t) \quad \text{as} \quad t \to \infty. \quad (2)$$

**Proof.** It follows from (1) that

$$\Phi_n(s) \sim n! / (s^{n+1} \hat{R}_0^n(s)). \quad (3)$$

Since $U(t)$ varies slowly as $t \to \infty$, $\hat{R}_0^n(s)$ also varies slowly as $s \to 0^+$. From this we have that $1/\hat{R}_0^n(s)$ varies slowly as $s \to 0^+$. Hence (see [3, Ch. 13.5])

$$\Phi_n(t) \sim t^n / U^n(t) \quad \text{as} \quad t \to \infty.$$ 

Theorem 1 is proved.

**Remark.** In the case $n = 1$, relation (2) coincides with the result of [4].

4. Assume that

$$a^{-1} = M X_1 < \infty. \quad (4)$$

In this case, from Theorem 1 we have that

$$\Phi_n(t) \sim (at)^n \quad \text{as} \quad t \to \infty.$$ 

It follows from (1) that

$$\Phi_n(s) = n! a^n / (s^{n+1} (1 - a s \hat{R}_1(s))^n).$$