Stable Subsets of Modules and the Existence of a Unit in Associative Rings

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Abstract. We obtain criteria for the existence of a (left) unit in rings (arbitrary, Artinian, Noetherian, prime, and so on) that are based on the systematic study of properties of stable subsets of modules and their stabilizers that generalize the technique of idempotents. We study a class of quasiunitary rings that is a natural extension of classes of rings with unit and of von Neumann (weakly) regular rings, which inherits many properties of these classes. Some quasiunitary radicals of arbitrary rings are constructed, and the size of these radicals "measures the probability" of the existence of a unit.

Key Words: quasiunitary module, quasiunitary ring, stable set of a ring, stabilizer, balanced set, idempotent, central idempotent, unit, left unit.

Introduction

Many properties of rings with unit (and unitary modules) can be extended to rings $R$ with $\sigma R = 0$ and $RR = R$ and to rings in which $r \in Rr$ for any $r \in R$ (the latter rings are called "left D-regular" in [1] and "left s-unital" in [2]; in this note we call them left quasiunitary or left QU rings). For instance, the polynomial ring over a left Noetherian left QU ring is Noetherian and left quasiunitary, an Artinian left quasiunitary ring is left Noetherian, a left QU ring without left ideals is a skew field, a left QU module over a left completely reducible ring (without unit) is completely reducible, quasiunitary left ideals of associative algebras are subspaces, etc.

The class of left QU-rings is much wider than those of rings with unit and is closed under the main constructions (Theorem 3.1). It is of interest to clarify under what additional assumptions, a left QU ring possesses a left (right) unit or is right quasiunitary. Criteria for the existence of a unit can be stated in a more general case in terms of local properties of certain subsets. To this end, we consider stabilizers of subsets and ballasts of rings and modules (Definitions 1.1 and 1.2), which are objects antipodal to annihilators but closely related to them.

The present note is an attempt to systematize some simple observations that generalize the technique of working with idempotents, on the base of an appropriate set of notions and notation. The abundance of assertions and corollaries makes the presentation locally trivial. The absence of a proof after a statement means that the proof can be obtained by direct calculation or that the statement is an immediate consequence of the preceding one. The main results are accumulated in §5, in Theorems 2.1–2.4, in Assertions 2.2 and 2.3, and in their corollaries. Example 5.1 answers a series of questions about the essence of the conditions of the theorems.

§1. Main notation, definitions, and examples

Everywhere below, $R$ and $T$ are rings, $RMT$ is a bimodule (in particular, a left module $M = RM$ for $T = \mathbb{Z}$ or for $T = \text{End}_R M$); we write $L < R M$ if $L$ is a submodule of $RM$ (and $L < RR$ means that $L$ is a left ideal of $R$), $L < R$ is an ideal, the symbol "\equiv" stands for equality by definition (by notation), \( \prod \) stands for the Cartesian product of rings, \( \boxplus \) for the direct product of rings (and for the direct sum of ideals), \( \oplus \) denotes the direct sum of modules, $C(R)$ is the center, $C^#(R)$ is the quasicenter (see Definition 1.5), $J(R)$ is the Jacobson radical, $\text{Soc}(R_R)$ and $\text{Soc}(RR_R)$ are the left and right socles of $R$ (the sum of all...

Translated from Matematicheskie Zametki, Vol. 61, No. 4, pp. 596–611, April, 1997.

Original article submitted December 9, 1994.
irreducible left (right) submodules), \(I(R), N(R),\) and \(U(R)\) are the sets of idempotent, nilpotent, and invertible elements of \(R,\) respectively, and \(QU(R)\) is the group of all quasiinvertible elements with the product \('\cdot'\): \(a * b \equiv a + b - ab.\) For \(A, B \subset R\) and \(N \subset RMT,\) \(C(A)\) is the centralizer of \(A\) in \(R\) and \(\ominus N\) and \(N^o\) are the left and right annihilators of \(N,\)

\[
A \cdot B \equiv \{a \cdot b \mid a \in A, b \in B\}, \quad A \cdot N \equiv \{an \mid a \in A, n \in N\},
\]

\[
AN \equiv \left\{ \sum a_i n_i \mid a_i \in A, n_i \in N \right\}, \quad \bar{A}N \equiv \{n - an \mid a \in A, n \in N\}.
\]

For \(r \in R\) and \(t \in T,\) the mappings of left and right translation

\[
\bar{r} : m \mapsto \bar{r}m \equiv m - rm, \quad \bar{t} : m \mapsto \bar{m}t \equiv m - mt
\]

are endomorphisms of the group \(M\), and the sets of all such endomorphisms are semigroups \((\bar{b} \circ \bar{a} = \bar{b} * \bar{a},\)

\(\bar{b} \circ \bar{a} = \bar{a} * \bar{b}));\) an element \(r\) is left quasiinvertible

\[
\iff r \in R\bar{r} \iff R\bar{r} = R \iff R \ast r = R; \text{ a left ideal } I \text{ is modular } \iff \exists r \in R: R\bar{r} \subset I.
\]

The notation \(S^A = S A, A^S = \bar{A}^S, S(A), A^F, F^A, A^F_M (A \subset RMT \text{ or } A \subset R), B_X(RM), B(RM), B(R),\)

and \(B(R)\) is introduced in Definitions 1.1-1.4, and the notation \(Q(X(RM), Q(RM) \equiv Q_R(RM), Q(R_R), Q(R), QL(R), QR(R),\) and \(QL(R)\) is defined in Corollary 7 of Theorem 2.2 and in Theorem 2.4. The rest of the terminology coincides with that of the monographs [3, 4].

**Definition 1.1.** By a left (right) stabilizer of a set \(A \subset RMT\) we mean the set

\[
S^A \equiv S A \equiv \{r \in R \mid rm = m \ \forall m \in A\} \quad (A^S \equiv A^S \equiv \{t \in T \mid mt = m \ \forall m \in A\}),
\]

and, for \(R = T,\) the set \(S(A) \equiv A^S \cap A^S\) is called the stabilizer of \(A.\) If \(S^A \neq \emptyset (A^S \neq \emptyset),\) then \(A\) is said to be left (right) stable.

Consider the dual object: for any \(A \subset R,\) introduce the sets of fixed points

\[
A^F \equiv \{m \in R \mid am = m \ \forall a \in A\} = \cap \text{Ker } \bar{a}, \quad F^A \equiv \{m \in R \mid ma = m \ \forall a \in A\} = \cap \text{Ker } \bar{a},
\]

\[
A^F_M \equiv \{m \in R \mid am = m \ \forall a \in A\}.
\]

**Definition 1.2.** The set \(B(RM) \equiv \cup \{F_M \mid r \in R\} (B(MT))\) of all left (right) stable elements of \(RMT\) is called the left (right) ballast of \(M, B(RMT) \equiv B(RM) \cap B(MT)\) is called the ballast of \(M,\) and the elements of \(B(RMT)\) are said to be stable (or \(s\)-elements).

**Definition 1.3.** A module \(RMT\) (a ring \(RRR\)) is said to be left (right) quasiunitary or a QU module if \(B(RM) = M (B(MT) = M),\) i.e., if \(m \in \bar{R}m\) for any \(m \in M.\) A ring \(R\) is said to be balanced if, for any \(r \in R,\) there exists \(s \in R\) such that \(sr = r = rs.\)

**Definition 1.4.** Let \(X \subset R.\) The subset of all \(X\)-stable elements

\[
B_X(RM) \equiv \{m \in R \mid \exists r \in X : rm = m\} \cup \{F_M \mid r \in X\}
\]

is called the \(X\)-ballast of \(R,\) and a module \(RMT\) (a ring \(R\)) is said to be quasiunitary (left or right quasiunitary) over \(X\) if

\[
B_X(M) = M \quad (B_X(RR) = R \text{ or } B_X(R) = R).
\]

For instance, \(R\) is left quasiunitary over a finite set \(X \subset R\) if and only if \(R\) has a left unit (Corollary 2 to Theorem 2.3).