HERMITE POLYNOMIAL ANSATZ FOR A MULTIDIMENSIONAL WELL

T. F. Pankratova

UDC 550.344

The Schrödinger operator in $\mathbb{R}^d$ with analytic potential that has a nondegenerate minimum (a well) at the origin is considered. Under the additional Diophantine condition on the frequencies, the full asymptotic expansions (as Planck's constant $h$ tends to zero) of a set of eigenfunctions (Ansatz with Hermite polynomials) and of eigenvalues with given quantum numbers ($n \in \mathbb{N}^d$, $|n| = 0, 1, 2, \ldots$) located at the bottom of the potential well are constructed in a neighborhood of the origin, which is independent of $h$. The asymptotics obtained can be prolonged onto a larger domain by using the ray method. A method of approximately describing the zero-sets of eigenfunctions (and of their intersections) is discussed. Some simple examples in the two-dimensional case are considered. Bibliography: 22 titles.

§ 1. INTRODUCTION

We consider the Schrödinger equation

$$-\frac{\hbar^2}{2} \Delta u + V u = Eu,$$

(1.1)

where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $V$ is a real-valued function defined on $\mathbb{R}^d$ that has a nondegenerate minimum at the origin $V(0) = 0$ (a well).

Let $V(x) > C > 0$ beyond some neighborhood of the minimum. As is known, in this case the spectrum of the self-adjoint Schrödinger operator $\mathcal{G}$ defined by the left-hand side of Eq. (1.1) is discrete [1]. We shall discuss the quasiclassical asymptotics ($h \to 0$) of the lower part of this spectrum.

It is of interest to consider the problem in the case where $V$ has a finite number of wells that are identical up to space translations and where the discrete spectrum of the operator $\mathcal{G}$ is as follows. There is a set of groups of eigenvalues on the axis. The number of eigenvalues in each group is equal to the number of wells. The distance between the groups is of order $h$. The distance between the eigenvalues within each group, i.e., the splitting amplitude, is exponentially small as $h \to 0$. The location of each group is determined by its quantum vector $n \in \mathbb{N}^d$, $|n| = 0, 1, 2, \ldots$.

These phenomena are investigated almost completely and rigorously in the one-dimensional case [2–12].

The case $d \geq 2$ is much more complicated [12–18]. It has been considered in quite a number of papers. Many results (on the level of theorems) are contained in the papers by Helffer and Sjöstrand ([12–14] and others, where the technique of pseudodifferential operators is used) and by Dobrokhotov, Kolokoltsov, and Maslov ([16–18] and others, where the tunnel canonical operator is applied). However, so far no effective (as in the case $d = 1$) asymptotic formula for the amplitude of the splitting of energy levels for arbitrary quantum vectors was obtained for problems with a number of identical wells. The methods used in [16–18] enable one to determine the splitting amplitude only for the lowest group ($n = 0$).

In this paper, for $d \geq 2$ we construct asymptotic series for a set of eigenvalues in a band $0 \leq E \leq E^* < C$ and for the corresponding eigenfunctions in a domain that is independent of $h$. Our approach is different from those mentioned above. The Hermite polynomial Ansatz is inspired by the harmonic oscillator [19]. The series constructed for eigenfunctions are new. They are formal expansions in powers of $h$ with coefficients that are analytic in a domain independent of $h$ if the potential is also. These series are divergent, but if we truncate them at $N$th terms, the sums of their remainders satisfy Eq. (1.1) with errors of order $h^{N+2} \cdot e^{-\frac{x}{\hbar}}$, where $S$ is a nonnegative function defined below. The above sums are the so-called

quasimodes [20], which satisfy the Schrödinger equation with exponentially small errors. The possibility of
constructing quasimodes in a sufficiently wide range and with an error smaller than the splitting amplitude
furnishes an opportunity to determine the asymptotic behavior of the eigenvalues and eigenfunctions with
sufficient accuracy (following the program executed for $d = 1$ in [10]). The series constructed can be used
to analyze the zero-sets of eigenfunctions in the case of a single well. Some examples of such an analysis
are provided in §5.

§ 2. ASYMPTOTIC EXPANSIONS FOR EIGENVALUES AND EIGENFUNCTIONS

For each quantum vector $n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$, we look for the value $E_n$:

$$E_n = \sum_{j=1}^{\infty} E_{nj} h^j$$

and the function $u_n(x)$:

$$u_n = \exp \left\{ - \frac{S^0}{h} \right\} \sum_{k=0}^{\infty} u_{nj} h^j$$

(where $S^0 = S^0(x)$, $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, and $u_{nj} = u_{nj}(x)$, $j = 0, 1, 2, \ldots$, are functions independent
of $h$) that formally satisfy Eq. (1.1).

We look for $S^0(x)$ in the form of half the sum of $d$ squared new unknown functions $\psi_1(x), \psi_2(x), \ldots, \psi_d(x)$, i.e.,

$$S^0(x) = \frac{1}{2} \sum_{i=1}^{d} \psi_i^2$$

on which the following orthogonality conditions are imposed:

$$\langle \nabla \psi_i, \nabla \psi_j \rangle = 0, \quad i \neq j$$

(the symbols $\nabla$ and $\langle \cdot, \cdot \rangle$ denote the gradient and the scalar product in $\mathbb{R}^d$, respectively).

The function $u_{n0}(x)$ is sought in the form

$$u_{n0}(x) = e^{P_n(x)} \prod_{i=1}^{d} H_{n_i} \left( \frac{\psi_i(x)}{\sqrt{h}} \right),$$

where $P_n(x)$ is a new unknown function and $H_{n_i}(t) = (-1)^{n_i} e^{t^2}(e^{-t^2})^{(n_i)}$ are the Hermite polynomials,
which satisfy the differential equation

$$H_{n_i}''(t) - 2t H_{n_i}'(t) + 2n_i H_{n_i}(t) = 0.$$  

We substitute the series (2.1) and (2.2) into (1.1) and equate the coefficients at each power of $h$ to zero.
Equating the coefficient at the zero power of $h$ to zero yields the eikonal equation for the function $S^0(x)$

$$(\nabla S^0)^2 = 2V,$$

which is equivalent to the system of $d$ nonlinear equations (2.4), (2.7) for $d$ unknown functions $\psi_1(x), \psi_2(x), \ldots, \psi_d(x)$, if one rewrites the left-hand side of (2.7) in terms of these functions using (2.3).

Equating the coefficient at the first power of $h$ to zero (while keeping in mind (2.4)-(2.7)) yields the
following equation for the function $P_n$ and the value $E_{n1}$:

$$\langle \nabla S^0, \nabla P_n \rangle = E_{n1} - \frac{\nabla S^0}{2} = \sum_{i=1}^{d} n_i (\nabla \psi_i)^2.$$