UNIFORM ASYMPTOTIC EXPANSIONS OF THE SOLUTIONS OF
STUECKELBERG’S SYSTEM WITH TURNING POINTS

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Uniform asymptotic expansions of the solutions of Stueckelberg’s system with analytic potentials in the vicinity
of the maximum of a potential barrier and in the vicinity of a potential-curve-crossing point are constructed.
Bibliography: 11 titles.

Stueckelberg’s systems, which arose in studying inelastic collisions of two atoms, are of the form
\begin{align*}
\begin{cases}
\hbar^2 u''_1 + \varphi_1 u_1 = \alpha u_2, \\
\hbar^2 u''_2 + \varphi_2 u_2 = \alpha u_1,
\end{cases}
\end{align*}

where \( \varphi_1, \varphi_2 \) and \( \alpha \) are analytic functions of \( z \), which varies in a bounded domain \( D \) that contains an
interval \([a, b]\) of the real axis, and of a parameter \( \delta \) in the domain \( G; \ h > 0 \) is a small parameter.

System (1) has been investigated by many authors [1–4], but in most publications only the model case
of the linear potentials \( \varphi_1 \) and \( \varphi_2 \) and constant coupling function \( \alpha \) was considered.

The comparison equation method [5–9] allows one to handle the problem for arbitrary analytic potentials
and an arbitrary coupling function.

1. Reduction of system (1) to a fourth-order equation and the properties of the symbol of
this equation

System (1) can be reduced to the fourth-order equation
\begin{align*}
&u'''' + \frac{1}{n^2} u''(\varphi_1 + \varphi_2) + \frac{1}{h^4}(\varphi_1 \varphi_2 - \alpha^2)u_1 \\
&+ 2\alpha \left( \frac{1}{\alpha} \right)' u''' + 2\alpha \left( \frac{\varphi_1}{\alpha} \right)' \frac{1}{h^2} u'_1 + \alpha \left( \frac{1}{\alpha} \right)'' u''' + \frac{1}{h^2} \alpha \left( \frac{u_1}{\alpha} \right)''' u_1 = 0
\end{align*}

for the function \( u_1 \), and it is this equation that is considered in what follows. The symbol of Eq. (2),
\( l(z, p) \equiv l(z, p, \delta, h) \), is of the form
\begin{align*}
l(z, p) &= p^4 + (\varphi_1 + \varphi_2)p^2 + (\varphi_1 \varphi_2 - \alpha^2) \\
&+ 2h\alpha \left( \frac{1}{\alpha} \right)' p^3 + 2h\alpha \left( \frac{\varphi_1}{\alpha} \right)' p + Ah^2 \left( \frac{1}{\alpha} \right)'' p^2 + h^2 \alpha \left( \frac{\varphi_1}{\alpha} \right)''.
\end{align*}

The roots of the characteristic equation
\( l(z, p) = 0 \)

depend upon the parameters \( \delta \) and \( h \), namely,
\( p_i = p_i(\delta, h) \quad (i = 1, 2, 3, 4). \)

For \( h = 0 \) we have
\begin{align*}
p_i(\delta, 0) &= (-1)^{i+1} \sqrt{\varphi(\delta) + \sqrt{\psi^2(\delta) + \alpha^2}} \equiv p_{i0}(\delta), \quad i = 1, 2, \\
p_j(\delta, 0) &= (-1)^{j+1} \sqrt{\varphi(\delta) - \sqrt{\psi^2(\delta) + \alpha^2}} \equiv p_{j0}(\delta), \quad j = 3, 4,
\end{align*}

where \( \varphi(\delta) = -\frac{1}{2}(\varphi_1 + \varphi_2), \psi(\delta) = \frac{1}{2}(\varphi_1 - \varphi_2). \)

Below, we intend to apply the method for constructing asymptotic expansions described in [10]. To this
end, we transform the symbol \( l(z, p) \) of Eq. (2) to a more convenient form by using the lemma below.

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Lemma 1. Let the symbol \( l(z, p) \equiv l(z, p, \delta, h) \) allow for a representation of the form

\[
l(z, p) = (p^2 + a_3 p + a_2)(p^2 + a_1 p + a_0) + c_3 p^3 + c_2 p^2 + c_1 p + c_0,
\]

where \( a_i, c_i \) (\( i = 0, 1, 2, 3 \)) are analytic functions of \((z, \delta) \in D \times G\). Let, in addition, \( c_i \) be analytic functions of \( h \) and \( c_i = O(h) \).

If the roots \( p_1 \) and \( p_2 \) of the equation \( p^2 + a_3 p + a_2 = 0 \) and the roots \( p_3 \) and \( p_4 \) of the equation \( p^2 + a_1 p + a_0 = 0 \) are distinct, i.e., \( p_i \neq p_k \) (\( i = 1, 2; k = 3, 4 \)), then there exist functions \( b_i, i = 0, 1, 2, 3 \), analytic for \((z, \delta) \in D \times G\) and \( h < \varepsilon \), such that

\[
l(z, p) = (p^2 + (a_3 + b_3)p + (a_2 + b_2))(p^2 + (a_1 + b_1)p + (a_0 + b_0)).
\]

**Proof.** Comparing the coefficients at \( p^n \) in (5) and (6), we see that \( b_i \) (\( i = 0, 1, 2, 3 \)) satisfy the equation

\[
M b + L b = c,
\]

where the vectors \( b \) and \( c \) and the operator \( L \) are defined by the equalities

\[
b = (b_3, b_2, b_1, b_0)^t, \quad c = (c_3, c_2, c_1, c_0)^t, \quad L b = (0, b_3 b_1, b_1 b_0 + b_2 b_1, b_2 b_0)^t,
\]

and the matrix \( M \) is of the form

\[
M = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & a_3 & 1 & 1 \\
a_0 & a_2 & a_1 & a_3 \\
0 & 0 & a_0 & a_2
\end{pmatrix}.
\]

Since \( c = h \tilde{c} \), we have \( b = h \tilde{b} \) and \( L b = h^2 L \tilde{b} \). The determinant

\[
det M = -(p_3 - p_1)(p_3 - p_2)(p_4 - p_1)(p_4 - p_2)
\]

of the matrix \( M \) does not vanish. Consequently, from Eq. (7) it follows

\[
\tilde{b} = M^{-1} \tilde{c} - h M^{-1} L \tilde{b}.
\]

For \( h \) small enough, in the bounded domain \( D \) the norm of the operator \( A = M^{-1} L \) satisfies the inequality \( \|A\| < 1 \), and, therefore, to solve Eq. (8), the method of successive approximations may be applied. For \((z, \delta) \in D \times G\) and \( h < \varepsilon \), the solution \( \tilde{b} \) is then represented in the form of a uniformly convergent series

\[
\tilde{b} = M^{-1} \tilde{c} - h A M^{-1} \tilde{c} - h^2 A^2 M^{-1} \tilde{c} - \ldots
\]

of analytic functions. Consequently, \( \tilde{b} \) is an analytic function of \( z, \delta, \) and \( h \).

**Lemma 2.** The function \( b \) can be represented as

\[
b = M^{-1}c + O(h).
\]

The proof immediately follows from the series expansion (9).

2. Potential barrier

We assume the functions \( \varphi_i(z) \) to be of the form \( \varphi_i(z) = \delta - V_i(z) \), where \( \delta \) is a parameter, \( \delta \in G \).

In case one of the functions \( V_i(z) \) has a maximum, we say that system (1) or Eq. (2) possesses a potential barrier.

Let the functions \( V_1(z), V_2(z), \) and \( \alpha(z) \) possess the following properties: