SINGULAR PARTS OF PLURIHARMONIC MEASURES

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A measure $\mu$ defined on the complex sphere $S$ is called pluriharmonic if its Poisson integral is a pluriharmonic function (in the unit ball of $\mathbb{C}^n$). A probability measure $\rho$ is called representing if $\int_S f \, d\rho = f(0)$ for all $f$ in the ball algebra. It is shown that singular parts of pluriharmonic measures and representing measures are mutually singular. Bibliography: 5 titles.

The basic definitions and notation used in the present paper are almost the same as in the monograph of W. Rudin [1]. In particular, $B = B_n = \{z \in \mathbb{C}^n : |z| < 1\}$ is the unit ball of $\mathbb{C}^n$, $S = \{\zeta \in \mathbb{C}^n : |\zeta| = 1\}$ is the unit sphere, $\sigma$ is the Lebesgue measure on $S$ normalized by the condition $\sigma(S) = 1$. By the definition, the ball algebra $A(B)$ consists of functions continuous in the closed ball $\overline{B}$ and holomorphic in $B$. Denote by $H(p, q)$ (for $(p, q) \in \mathbb{Z}_+^2$) the space of all harmonic homogeneous polynomials in $\mathbb{C}^n$ of degree $p$ in variables $z_1, \ldots, z_n$ and of degree $q$ in $\overline{z}_1, \ldots, \overline{z}_n$. $M(S)$ denotes the space of regular complex Borel measures on $S$. It is necessary to note that $P[\mu]$ denotes the ordinary (not $A_4$-harmonic as in [1]) Poisson integral of $\mu \in M(S)$.

Definition. A probability measure $\rho \in M(S)$ is called representing (the value at the point 0) if $\int_S f \, d\rho = f(0)$ for all $f \in A(B)$. The set of all representing measures is denoted by $M_0(S)$.

Definition. A measure $\mu \in M(S)$ is called pluriharmonic if its Poisson integral $P[\mu]$ is a pluriharmonic function. Denote by $PM(S)$ the set of all pluriharmonic measures. By the definition, the set $PM^s(S)$ consists of singular parts of pluriharmonic measures.

Let $\rho \in M_0(S)$ and let $\mu^s$ be the singular part of a positive measure $\mu \in PM(S)$. It is proved in [2] that the measures $\mu^s$ and $\rho$ are mutually singular in this case. At the same time, A. B. Aleksandrov posed the question (see [2], Chap. 5, 3.3.3) of whether the same statement is still true for all measures $\mu^s \in PM^s(S)$.

A positive answer to this question is given in the present paper (Theorem 10).

First of all, we obtain some generalizations of the theorems of Valskii and Henkin.

Let $X = X(S)$ be an arbitrary closed subspace of the continuous functions space $C(S)$. Let us extend elements of $X$ up to functions harmonic in the ball (by means of the Poisson integral). We denote the space obtained by $X(B)$.

Definition. A sequence of functions $\{f_j\}_{j=1}^\infty$ is called an $X$-sequence (a Montel sequence for a fixed space $X$) if

1. $f_j \in X(B)$ and the sequence $\{f_j\}$ is uniformly bounded on $\overline{B}$;
2. $f_j(z) \to 0$ as $j \to \infty$ for all $z \in B$.

Remark 1. It is not difficult to see that every Montel sequence converges to zero uniformly on compact subsets of the ball $B$.

Definition. A measure $\mu \in M(S)$ is called an $X^\perp$-measure (a Henkin measure) if $\lim_{j \to \infty} \int_S f_j \, d\mu = 0$ for every $X$-sequence $\{f_j\}_{j=1}^\infty$.

It follows immediately from the definition that the class of all $X^\perp$-measures is norm-closed in $M(S)$. If $\mu \in M(S)$ and $\int_S f \, d\mu = 0$ for all $f \in X$, then we write $\mu \in X^\perp$. Note that every measure $\mu \in X^\perp$ is a Henkin measure. The following lemma shows that all measures that are absolutely continuous with respect to $\sigma$ are also Henkin measures.

Lemma 2. Let \( g \in L^1(\sigma) \). Then \( \sigma \) is an \( X^\perp \)-measure.

Proof. One can approximate the function \( g \) by harmonic polynomials. On the other hand, the space of Henkin measures is closed; thus it is enough to consider the case \( g \in H(p,q) \), \((p,q) \in \mathbb{Z}_+^2 \). Let \( \{f_j\}_{j=1}^{\infty} \) be a Montel sequence. Denote by \( h_j \) the projection of the function \( f_j \) on the space \( H(q,p) \). The mutual orthogonality of spaces \( H(r,s) \) and the Hölder inequality imply the inequality

\[
\left| \int_S f_j g \, d\sigma \right| = \left| \int_S h_j g \, d\sigma \right| \leq ||h_j||_2 ||g||_2. \tag{1}
\]

Now fix \( r \in (0,1) \). One has \( f_j(r\zeta) = h_j(r\zeta) + P[f_j - h_j](r\zeta) \) for \( \zeta \in S \). The summands on the right-hand side of the last equality are mutually orthogonal in \( L^2(\sigma) \); therefore, by Remark 1 we obtain \( ||h_j||_2 = r^{-p-q}||(h_j)_r||_2 \to 0 \). By (1), this proves the lemma.

Remark 3. It follows from Lemma 2 that every Montel sequence \( \{f_j\}_{j=1}^{\infty} \) converges to zero in the weak topology of the space \( L^2(\sigma) \) as \( j \to \infty \).

Given a measure \( \mu \in M(S) \), denote by \( ||\mu||_{X^*} \) the norm of the functional on \( X \) generated by \( \mu \). Define \( f_\sigma(\zeta) = f(r\zeta) \) for \( \zeta \in S \). We assume in what follows that the condition \( f \in X(B) \) implies \( f_\sigma \in X \) for \( r \in (0,1) \).

The following theorem is proved in [3] for \( X = A(S) \) (see also [1], 9.2).

Theorem 4 (theorem of Valskii). Let \( \mu \) be an \( X^\perp \)-measure. Take \( \varepsilon > 0 \). Then there exists a function \( g \in L^1(\sigma) \) such that \( \mu - g\sigma \in X^\perp \) and \( ||g||_1 \leq ||\mu||_{X^*} + \varepsilon \).

By Remark 1 and Lemma 2, to prove this theorem one can repeat the argument of Valskii.

Let \( \Omega \subset \mathbb{Z}_+^2 \) and let \( Y \) denote one of the spaces \( C(S) \) or \( L^2(S) \). Denote by \( Y_\Omega \) the closure in \( Y \) of the linear span of all spaces \( H(p,q) \) with \((p,q) \in \Omega \).

Denote by \( C_\Omega \) the orthogonal projection from \( L^2(S) \) onto \( L^2_\Omega(S) \).

By the definition, a Hankel operator with symbol \( \varphi \in C(S) \) is the operator \( V_{\Omega,\varphi} : L^2(S) \to L^2(S) \) given by the formula

\[
V_{\Omega,\varphi}[f] = \varphi \cdot C_\Omega[f] - C_\Omega[\varphi].
\]

Take \( \Omega_1, \Omega_2 \subset \mathbb{Z}_+^2 \) such that \( \Omega_1 \cap \Omega_2 = \emptyset \). It follows from the definition that the equality \( V_{\Omega_1,\varphi} + V_{\Omega_2,\varphi} = V_{\Omega_1 \cup \Omega_2,\varphi} \) holds in this case. Define \( \Lambda \triangleq \{(p,q) \in \mathbb{Z}_+^2 : pq \neq 0\} \) and \( \Gamma \triangleq \mathbb{Z}_+^2 \setminus \Lambda \). Note that \( C_\Lambda(S) \) coincides with the set of functions \( f \in C(S) \) such that \( \int_S g \, d\mu = 0 \) for all \( \mu \in PM(S) \). In other words, \( C_\Lambda(S)^\perp = PM(S) \). Since \( V_{\mathbb{Z}_+^2,\varphi} \equiv 0 \), we have \( V_{\Lambda,\varphi} \equiv -V_{\Gamma,\varphi} \). Define \( \Gamma_0 = \{(0,0)\}, \Gamma_1 = \{(p,0) : p \in \mathbb{Z}_+\} \), \( \Gamma_2 = \{(0,q) : q \in \mathbb{Z}_+\} \); then \( V_{\Gamma_0,\varphi} = V_{\Gamma_1,\varphi} + V_{\Gamma_2,\varphi} - V_{\Gamma_0,\varphi} \). Note that \( V_{\Gamma_1,\varphi} \) is a classical Hankel operator, the properties of the operators \( V_{\Gamma_2,\varphi} \) and \( V_{\Gamma_1,\varphi} \) are similar, and the operator \( V_{\Gamma_0,\varphi} \) has image of finite dimension. Therefore, the following theorem is valid.

Theorem 5. (see [1], 6.5.4) Let \( \varphi \in C^1(S) \). Then \( V_{\Lambda,\varphi}(C(S)) \subset C(S) \) and the operator \( V_{\Lambda,\varphi} : C(S) \to C(S) \) is compact.

According to Henkin, \( A(S)^\perp \)-measures are called \( A \)-measures. The following theorem is proved in [4] for \( A \)-measures (see also [1], 9.3).

Theorem 6 (theorem of Henkin). Let \( \lambda \) be a \( C_\Lambda(S)^\perp \)-measure and \( \mu \ll \lambda \). Then \( \mu \) is a \( C_\Lambda(S)^\perp \)-measure.

Proof. The conditions of the theorem imply that \( \mu = \varphi \lambda \) for some function \( \varphi \in L^1(||\lambda||) \). Since the set of Henkin measures is closed in \( M(S) \), we assume that \( \varphi \in C^1 \). By the theorem of Valskii and by Lemma 2, it is enough to prove that

\[
\lim_{j \to \infty} \int_S f_j \varphi \, d\nu = 0 \tag{2}
\]

for \( \nu \in C_\Lambda(S)^\perp, \varphi \in C^1(S) \), and for an arbitrary \( C_\Lambda(S)^\perp \)-sequence \( \{f_j\}_{j=1}^{\infty} \).