Let $F$ be a compact subset of $\mathbb{C}$, let $\mu$ be a Borel measure on $F$, and let $\rho(z)$ be the distance of $z$ to $F$. Denote

$$A_K(f)(z) = \int_F K(\zeta, z) f(\zeta) \, dm(\zeta), \quad z \in \mathbb{C} \setminus F,$$

where $K(\zeta, z)$ is either $(\zeta - z)^2$ or $|\zeta - z|^{-1}$, and $m$ is the Lebesgue measure. Let $\psi$ be a monotone nondecreasing positive function on $(0, \infty)$ and let $\Phi(z) = \psi(\rho(z)) \rho(z)$, $z \in \mathbb{C} \setminus F$. Under some additional assumptions on $\mu$, it is proved that $A_K$ is bounded from $L^2(\mu)$ to $L^2(\Phi m)$ if and only if

$$\int_0^1 \frac{\psi(t)}{t} \, dt + \int_1^{+\infty} \frac{\psi(t)}{t^2} \, dt < \infty.$$

Thus, no interference of values of $K$ of various signs is observed in such a situation. Bibliography: 4 titles.

0. Let $\mu$ be a Borel measure supported on a compact set $F \subset \mathbb{C}$ and satisfying the Carleson condition. In this paper, we study the behavior of the Calderón-Zygmund operators $A_K$ with the kernels $K(\zeta, z) = (\zeta - z)^{-2}$ and $K(\zeta, z) = |\zeta - z|^{-1}$

$$A_K(f)(z) = \int_F K(\zeta, z) f(\zeta) \, d\mu(\zeta), \quad z \in \mathbb{C} \setminus F.$$

We are interested in the conditions on a monotone nondecreasing positive function $\psi$ satisfying $\int_1^{+\infty} \frac{\psi(t)}{t^2} \, dt < \infty$ that guarantee the boundedness of $A_K$ regarded as an operator from $L^2(\mu)$ to $L^2(\Phi m)$, where $m$ is the Lebesgue measure, $\Phi(z) = \rho(z) \psi(\rho(z))$ ($z \in \mathbb{C} \setminus F$), and $\rho$ is the distance to $F$.

In the case of the standard one-dimensional Cantor set $F$, one-dimensional Hausdorff measure $\mu$ on $F$, $K(\zeta, z) = (\zeta - z)^{-2}$, and $\psi \equiv 1$, the question of whether $A_K$ acts from $L^2(\mu)$ to $L^2(\Phi dm)$, was posed by E. M. Dyn'kin in the survey [2]. In turns out (see Theorem 3) that in this case $A_K(L^2(\mu)) \subset L^2(\Phi dm)$ if and only if

$$\int_0^1 \frac{\psi(t)}{t} \, dt < +\infty. \quad (0)$$

The same condition on $\psi$ is equivalent to the inclusion $A_K(L^2(\mu)) \subset L^2(\Phi dm)$ if $\mu$ satisfies (5) and $K(\zeta, z) = |\zeta - z|^{-1}$; see Theorem 4. By Theorem 1, inequality (0) implies the inclusion $A_K(L^2(\mu)) \subset L^2(\Phi m)$ for the kernel $K(\zeta, z) = |\zeta - z|^2$ as well. This means that the interference phenomenon is entirely absent in the above cases.

1. Let $\mu$ be a Borel measure on $\mathbb{C}$ with compact support $F = \text{supp} \mu$. Assume that $\mu$ satisfies the Carleson condition

$$\mu(B(z, r)) \leq c_1 r, \quad z \in \mathbb{C}, \quad r > 0.$$
Here and in what follows $B(z, r) = \{ \zeta \in \mathbb{C} : |\zeta - z| < r \}$. We consider integral operators of the form

$$A_K(f)(z) = \int_F K(\zeta, z) f(\zeta) d\mu(\zeta), \quad f \in L^2(\mu), \quad z \in CF$$

($CF = C \setminus F$) with kernels $K : F \times CF \to \mathbb{C}$ that satisfy the condition

$$|K(\zeta, z)| \leq c_2 |\zeta - z|^{-2}, \quad \zeta \in F, \quad z \in CF. \quad (1)$$

Further, let $\psi$ be a monotone nonincreasing positive function on $(0, +\infty)$ such that

$$\int_{1}^{+\infty} \frac{\psi(t)}{t^2} dt < +\infty.$$

Denote by $\rho(z)$ the distance from $z \in \mathbb{C}$ to $F$ and put $\Phi(z) = \psi(\rho(z)) \rho(z)$, $z \in CF$.

We study conditions on $\psi$ which guarantee that $A_K$ acts from $L^2(\mu)$ to $L^2(\Phi m)$ ($m$ is the Lebesgue measure). By the closed graph theorem, if $A_K$ acts in the above way, then $A_k$ is a fortiori bounded, i.e., $A_K \in \mathcal{L}(L^2(\mu), L^2(\Phi m))$. Now, we define

$$B_{K, \psi}(f)(z) = \Phi^{1/2}(z) A_K(f)(z), \quad z \in CF, \quad f \in L^2(\mu).$$

We formulate some conditions on $F$ under which $B_{K, \psi}$ is of weak type $(2, 2)$. Observe that $B_{K, \psi} \in \mathcal{L}(L^2(\mu), L^2(\Phi m))$ if and only if $A_K \in \mathcal{L}(L^2(\mu), L^2(\Phi m))$, and, moreover, $\|B_{K, \psi}\| = \|A_K\|$.

2. Theorem 1. Let

$$\int_{0}^{1} \frac{\psi(t)}{t} dt < +\infty. \quad (2)$$

Then $A_K \in \mathcal{L}(L^2(\mu), L^2(\Phi m))$.

Proof. Let $f \in L^2(\mu)$, $g = A_K(f)$. By (1) and the Cauchy–Bunyakovskii inequality, we obtain

$$|g(z)|^2 \leq c_2^2 \int_F \frac{|f(\zeta)|^2 \psi(|\zeta - z|)}{|\zeta - z|^2} d\mu(\zeta) \int_F \frac{d\mu(\zeta)}{\psi(|\zeta - z|)|\zeta - z|^2}, \quad z \in CF.$$

Now, since $\mu$ is a Carleson measure and $\psi$ is a monotone nonincreasing function, we can write

$$\int_F \frac{d\mu(\zeta)}{\psi(|\zeta - z|)|\zeta - z|^2} \leq \frac{1}{\psi(\rho(z))} \int_F \frac{d\mu(\zeta)}{|\zeta - z|^2} \leq \frac{2c_1}{\psi(\rho(z)) \rho(z)},$$

$$|g(z)|^2 \Phi(z) \leq 2c_1 c_2^2 \int_F \frac{|f(\zeta)|^2 \psi(|\zeta - z|)}{|\zeta - z|^2} d\mu(\zeta) \quad z \in CF. \quad (3)$$

For $A > 0$ we put $G_A = \bigcup_{z \in F} B(z, A)$. Then

$$\int_{G_A} |g(z)|^2 \Phi(z) dm(z) \leq 2c_1 c_2^2 \int_F |f(\zeta)|^2 d\mu(\zeta) \int_{G_A} \frac{\psi(|\zeta - z|)}{|\zeta - z|^2} dm(z) \leq 2c_1 c_2^2 \|f\|^2_{L^2(\mu)} \int_{0}^{a} \frac{\psi(t)}{t} dt.$$

Here for $a$ we may take $A + \text{diam } F$. 1803