A STRUCTURAL DESCRIPTION OF FUNCTIONS THAT ARE DEFINED ON A PLANE CONVEX DOMAIN AND HAVE THE PRESCRIBED RATE OF APPROXIMATION BY ALGEBRAIC POLYNOMIALS

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Dedicated to L. D. Faddeev
on the occasion of his 60th birthday

We fix $0 < \alpha$, $0 < p \leq \infty$, and a positive integer $m$. Let $f$ be a function defined on a plane convex domain $G$ and let $E_m(f, L^p(G))$ denote the best approximation of $f$ in $L_p(G)$ by algebraic polynomials of degree $m$. A description of the functions $f \in L_p(G)$ satisfying the inequalities

$$E_m(f, L^p(G)) \leq C m^{-\alpha}, \quad m = 1, 2, \ldots,$$

is given. Bibliography: 7 titles.

§ 1. INTRODUCTION

One of the main problems in the approximation theory is the description (in intrinsic terms) of classes of functions with a given rate of approximation by functions from a model class, such as splines, rational functions, trigonometric or algebraic polynomials. In the present paper we give a structural description of functions that are defined on a bounded convex set $F$ and have the property

$$E_m(f, L_p(F)) = O(m^{-\alpha}) \quad \text{as} \quad m \to \infty,$$

(1)

where $0 < p \leq \infty$, $0 < \alpha$. Here

$$E_m(f, L_p(F)) = \inf \|f - \pi\|_{L_p(F)},$$

with the infimum taken over algebraic polynomials of order no greater than $m$.

Throughout the paper we use the following notation:

$n$ is the dimension of the space $\mathbb{R}^n$;

$\mu_n$ is the Lebesgue measure in $\mathbb{R}^n$;

$F$ is a subset of $\mathbb{R}^n$ measurable in the sense of Lebesgue;

$L_p(F, \mu_n) = L_p(F)$, $0 < p \leq \infty$, is the Lebesgue space, i.e., the set of functions $f$ with domain $F$ such that

$$\|f\|_{L_p(F)} = \left( \int_F |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty,$$

(for $p = \infty$ the integral on the right-hand side should be replaced by the essential supremum, as usual);

$P_m$, where $m \in \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$, is the set of all algebraic polynomials in $n$ variables of order no greater than $m$, i.e., the set of functions $\pi(x)$ admitting the representation

$$\pi(x) = \sum b_\beta \, x^\beta = \sum b_\beta x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n},$$

where $b_\beta \in \mathbb{R}$ and the summation is taken over all multi-indices $\beta = (\beta_1, \ldots, \beta_n)$ with nonnegative integral entries such that $|\beta| = \beta_1 + \ldots + \beta_n \leq m$;

$D(x, a)$ is the open ball of radius $a > 0$ centered at the point $x \in \mathbb{R}^n$;

\[ \partial F \text{ is the boundary of the set } F; \]
\[ |T| \text{ is the number of elements of the set } T; \]
\[ \alpha > 0, \ 0 < \theta, \text{ and } p \leq \infty \text{ are certain parameters.} \]

Before stating the results of the paper, we shall give several remarks of a historical nature.

In the most general setting, the problem considered has the following form:

Describe in intrinsic terms (e.g., using the modules of continuity or locally polynomial mappings; note that systematically the use of the latter was started by Yu. A. Brudnyi) the set of functions \( f \in L_p(F) \) satisfying the condition

\[
\left( \sum_{m=0}^{\infty} \left( E_2^n(f, L_p(F)) 2^{m\alpha} \right)^{1/\theta} \right)^{1/\theta} < \infty
\]

and, in particular, satisfying condition (1) (which corresponds to the case \( \theta = \infty \)). In what follows, the functional space introduced above is denoted by \( E_{\alpha, p, \theta}^\infty(F) \).

At present time, little is known about the space \( E_{\alpha, p, \theta}^\infty(F) \). There are many papers, going back to the results of S. N. Bernshtein, which are devoted to the well-known one-dimensional case, \( n = 1, \ E = (0, 1) \). Here we should mention papers by S. M. Nikol'skii, V. K. Dzyadyk, A. F. Timan, G. G. Lorentz, Brudnyi, R. Devore, Z. Ditzian, E. M. Dyn'kin, M. K. Potapov, and V. Totik (see [1–3]; this list is on no account complete). Except for this case and the cases that can be solved by elementary arguments (e.g., those using the construction of the Cartesian product) the author knows only three relevant results.

(a) \( F \) is a homogeneous space, \( p = \infty \) (Ragozin [4]);

(b) \( F \) is an \((n - 1)\)-dimensional algebraic surface of a specific form (Nikol'skii [5]; note, that the case where \( F \) is the \((n - 1)\)-dimensional sphere was considered earlier in papers by G. G. Kushnirenko, Pavelke, Nikol'skii, and P. I. Lizorkin);

(c) \( F \) is the \( n \)-dimensional ball (the corresponding result can be derived from the previous ones).

Before writing this paper, the author knew about only one method (essentially due to Bernshtein), allowing one to describe the spaces \( E_{\alpha, p, \theta}^\infty(G) \), where \( G \) is an open subset in \( \mathbb{R}^n \). To illustrate it we use the ball \( D(0, 1) \).

**Theorem B.** Let \( S^n \) be the \( n \)-dimensional sphere of unit radius in \( \mathbb{R}^{n+1} \). Then the space \( E_{\alpha, p, \theta}^\infty(D(0, 1)) \) consists of functions \( f = f(x_1, \ldots, x_n) \) such that the restriction of the function \( g(x_1, \ldots, x_n, x_{n+1}) = f(x_1, \ldots, x_n) \) onto \( S^n \) belongs to \( E_{\alpha, p, \theta}^\infty(S^n) \). \( \Box \)

Let us state the above-mentioned result of Kushnirenko.

**Theorem K.** \( E_{\alpha, p, \theta}^\infty(S^n) = \text{Lip}^\alpha(S^n), \ 0 < \alpha < 1. \)

Note that for \( 0 < \alpha < 1 \) the results of Ragozin [4] and Nikol'skii [5] can be stated in a quite similar way.

The following result is an easy corollary of Theorems K and B.

**Proposition 1.** A function \( f \in L_\infty(D(0, 1)) \) belongs to the space \( E_{\alpha, p, \theta}^\infty(D(0, 1)) \), \( 0 < \alpha < 1 \), if and only if the following two conditions are fulfilled:

(a) for every two points \( x, y \in D(0, 1) \) we have the inequality

\[
|f(x) - f(y)| \leq c \left( \frac{|x - y|}{\max(\sqrt{1 - |x|}, \sqrt{1 - |y|})} \right)^\alpha,
\]

(b) for every two points \( x, y \in D(0, 1) \) with \( |y| = |x| \), we have \( |f(x) - f(y)| \leq c|x - y|^\alpha. \) \( \Box \)

However, it is easy to see that the above-described technique (which is due to Bernshtein) is applicable only to a rather restricted class of domains. In particular, it is required that the boundary of the domain \( G \) be an algebraic surface. Our approach here is based on somewhat different ideas, and it allows us to describe the space \( E_{\alpha, p}^\infty(G) \) for a wider class of domains, including \( n \)-dimensional bounded domains with the boundary of class \( C^2 \), two-dimensional bounded convex sets, and, possibly, some others.

Let us briefly explain the essence of our approach and the subsequent content of the paper. Note that the proofs of the assertions stated below are not given. The author is going to continue the study of the