SMOOTH AND CONVERGENT $\varepsilon$-APPROXIMATIONS OF THE FIRST BOUNDARY-VALUE PROBLEM FOR EQUATIONS OF THE KELVIN-VOIGHT AND OLDROYD FLUIDS

A. P. Oskolkov

UDC 517.9

Dedicated to L. D. Faddeev on the occasion of his 60th birthday

Smooth convergent $\varepsilon$-approximations (11)-(13) for the equations of Oldroyd (8) and Kelvin-Voight (9), (10) fluids are constructed. It is shown that the first initial boundary-value problem for two-dimensional system (11) and three-dimensional systems (12) and (13) for every $\varepsilon > 0$ has a unique classical solution, and as $\varepsilon \to 0$ these solutions converge to classical solutions of the first initial boundary-value problem for Eqs. (8), (9), and (10) respectively. Bibliography: 10 titles.

1. When numerically solving equations of motion of viscous incompressible fluids, it is expedient [1, 2] to get rid of the incompressibility condition $\text{div} \, v = 0$, approximating these equations by perturbed parabolic (or pseudoparabolic) quasilinear systems with a small parameter $\varepsilon > 0$. One such $\varepsilon$-approximation for the Navier-Stokes equation

\[ \frac{\partial v}{\partial t} - \nu \Delta v + v_k v_{x_k} + \nabla p = f, \quad \text{div} \, v = 0 \]  

has the form [1, 3, 4]

\[ \frac{\partial v^\varepsilon}{\partial t} - \nu L^\varepsilon v^\varepsilon + M^\varepsilon(v^\varepsilon) = f, \quad \varepsilon > 0, \]  

where

\[ L^\varepsilon v^\varepsilon := \Delta v^\varepsilon + \frac{1}{\varepsilon} \text{grad div} \, v^\varepsilon, \quad \varepsilon > 0, \]  

is a linear elliptic operator and

\[ M^\varepsilon(v^\varepsilon) := v_k^\varepsilon v_{x_k}^\varepsilon + \frac{1}{2} v^\varepsilon \text{div} \, v^\varepsilon \]  

are the penalized nonlinear terms.

The first initial boundary-value problem for Eqs. (2)-(4),

\[ v^\varepsilon|_{t=0} = v_0(x), \ x \in \Omega; \ v^\varepsilon|_{\partial \Omega} = 0, \ t \in \mathbb{R}^+: = [0, \infty), \]  

is studied in [1, 3-5]. It is shown there that if

\[ \Omega \subset \mathbb{R}^2, \ \partial \Omega \in C^2; \ v_0(x) \in \mathcal{J}_2^2(\Omega); f, f_t \in L_2(Q_{\infty}) \]  

(throughout the paper we use the notation from [1]), then the $\varepsilon$-approximation (2)-(4) of the first initial boundary-value problem (1), (5) for the Navier-Stokes equation in the two-dimensional case ($\Omega \subset \mathbb{R}^2$) is smooth and convergent. This means that for every $\varepsilon > 0$ the perturbed initial boundary-value problem (2)-(4), (5) has a unique classical solution $v^\varepsilon$ such that

\[ v_{x_k}^\varepsilon, v_t^\varepsilon \in C(\mathbb{R}^+; L_2(\Omega)); \ v_{x_k}^\varepsilon, v_{x_t}^\varepsilon \in L_2(Q_{\infty}), \ Q_{\infty} := \Omega \times \mathbb{R}^+, \]  

and as $\varepsilon \to 0$ the solutions $\{v^\varepsilon\}$ converge to the unique classical solution $(v, p)$ of the first initial boundary-value problem (1)-(5) for the Navier-Stokes equation.
In the present paper we suggest smooth and convergent $\varepsilon$-approximations for the first initial boundary-value problem for equations of the Oldroyd fluids of order $L = 1, 2, \ldots$ [5, 6],

$$v_t - \nu \Delta v_t + v_k v_{z_k} - \sum_{l=1}^{L} \beta_l \Delta u_l + \nabla p = f, \ \text{div} \ v = 0,$$

$$v_{tt} + \alpha_l u_l = v, \ l = 1, \ldots, L; \nu > 0; \alpha_l, \beta_l > 0, l = 1, \ldots, L,$$

for equations of the Kelvin–Voight fluids of order $L = 0$ [5–7],

$$v_t - \kappa \Delta v_t - \nu \Delta v_t + v_k v_{z_k} + \nabla p = f, \ \text{div} \ v = 0, \ k, \kappa > 0,$$

and equations of the Kelvin–Voight fluids of order $L = 1, 2, \ldots$ [5–7],

$$v_t - \kappa \Delta v_t - \nu \Delta v_t - \sum_{l=1}^{L} \beta_l \Delta u_l + v_k v_{z_k} + \nabla p = f, \ \text{div} \ v = 0,$$

$$u_{tt} + \alpha_l u_l = v, \ l = 1, \ldots, L; \nu, \kappa > 0; \alpha_l, \beta_l > 0, l = 1, \ldots, L.$$

(10)

These $\varepsilon$-approximations respectively have the form

$$v_\varepsilon - \nu L^{\varepsilon} v_\varepsilon - \sum_{l=1}^{L} \beta_l L^{\varepsilon} u_{\varepsilon} + M^{\varepsilon}(v_\varepsilon) = f, \ u_{\varepsilon} + \alpha_l u_{\varepsilon} = v_{\varepsilon}, \ l = 1, \ldots, L;$$

$$Z(v_\varepsilon) = v_\varepsilon - \kappa L^{\varepsilon} v_\varepsilon - \nu L^{\varepsilon} v_\varepsilon + M^{\varepsilon}(v_\varepsilon) = f;$$

$$v_\varepsilon - \kappa L^{\varepsilon} v_\varepsilon - \nu L^{\varepsilon} v_\varepsilon - \sum_{l=1}^{L} \beta_l L^{\varepsilon} u_{\varepsilon} + M^{\varepsilon}(v_\varepsilon) = f, \ u_{\varepsilon} + \alpha_l u_{\varepsilon} = v_{\varepsilon}, \ l = 1, \ldots, L.$$

(13)

Pseudoparabolic systems (12) and (13) regularize parabolic system (2) as Eqs. (9) of the Kelvin–Voight fluids and Eq. (10) of the Kelvin–Voight fluids of order $L = 1, 2, \ldots$ regularize the Navier–Stokes equation [6, 7].

The first initial boundary-value condition for systems (8), (10), and (13) is posed as follows:

$$v|_{t=0} = v_0(x), \ u|_{t=0} = u_0(\xi) \in \Omega; \quad v|_{\Omega_t = 0} = u|_{\Omega_t = 0} = 0, \ t \in \mathbb{R}^+; \ l = 1, 2, \ldots, L.$$

(14)

The main results of this paper are formulated in the following two theorems.

**Theorem 1.** Let condition (6) hold. Then $\forall \varepsilon > 0$ the initial boundary-value problem (11), (14) has a unique solution $v_\varepsilon$ satisfying (7).* This solution obeys the inequality

$$\|v_\varepsilon, v_t\|_{C(\mathbb{R}^+; L_2(\Omega))} + \frac{1}{\varepsilon} \|\text{div} v_\varepsilon\|_{C(\mathbb{R}^+; L_2(\Omega))} + \|v_{\varepsilon x}, v_{\varepsilon t}\|_{L^2(Q_\infty)} + \frac{1}{\varepsilon} \|v_{\varepsilon t}\|_{L^2(Q_\infty)} + \frac{1}{\varepsilon^2} \|\text{grad} \ v_\varepsilon\|_{L^2(Q_\infty)}$$

$$\leq C(v^{-1}, \|v_0\|_{L^2(\Omega)}, \|f, f_t\|_{L^2(Q_\infty)}).$$

(15)

For $\varepsilon \to 0$ we have the passages to the limit

$$v_\varepsilon \to v_\xi, \ v_t \to v_t \text{ in } C(\mathbb{R}^+; L_2(\Omega)),$$

$$\text{div} v_\varepsilon \to 0 \text{ in } C(\mathbb{R}^+; L_2(\Omega)),$$

$$v_{\varepsilon x}, v_{\varepsilon t} \to v_{xx}, v_{xt} \text{ in } L_2(Q_\infty),$$

$$-\frac{1}{\varepsilon} \text{div} v_{\varepsilon t} \to p(x, t) \text{ in } L_2(\mathbb{R}^+; W_2^1(\Omega)),$$

(19)

and $(v, \{u_l\}, p)$ is the unique solution of the first initial boundary-value problem (8), (14) for equations of the Oldroyd fluids of order $L = 1, 2, \ldots$ such that

$$v_\xi, v_t \in C(\mathbb{R}^+; L_2(\Omega)); \ v_{xx}, v_{xt} \in L_2(Q_\infty); \ \nabla p \in L_2(Q_\infty),$$

$$\|v_{\xi x}, v_{\xi t}\|_{C(\mathbb{R}^+; L_2(\Omega))} + \|v_{\xi t}, v_{\xi xx}\|_{L^2(Q_\infty)} + \|\nabla p\|_{L^2(Q_\infty)} \leq C_1.$$  

(20)

The existence of such a solution to problem (8), (14) was proved by the present author in [6].

* The functions $\{u_{\varepsilon t}\}$ in problems (11), (4) and (13), (4) play an auxiliary role and their properties are completely determined by those of the function $v_\varepsilon$. 

1716