ON AN INITIAL BOUNDARY-VALUE PROBLEM FOR THE EQUATION OF MAGNETOHYDRODYNAMICS WITH THE HALL AND ION-SLIP EFFECTS

G. Mulone and V. A. Solonnikov

Dedicated to N. N. Uraltseva on her jubilee

This paper is concerned with the three-dimensional initial boundary-value problem for the equations of magnetohydrodynamics with additional nonlinear terms stemming from a more general relationship between the electric field and the current density. The problem governs the motion of a viscous incompressible conducting liquid in a bounded container with an ideal conducting surface. The existence of a solution which is close to a certain basic solution is proved. The solution is found in the anisotropic Sobolev spaces $W^{2,1}_p$ with $p > 5/2$. The proof relies on the theory of general parabolic initial boundary-value problems. Bibliography: 16 titles.

§ 1. INTRODUCTION

Motion of a viscous incompressible conducting fluid in a magnetic field is governed by a system consisting of the Navier-Stokes and the Maxwell equations,

$$
\frac{\partial \vec{v}}{\partial t} - \nu \Delta \vec{v} + (\vec{v} \cdot \nabla)\vec{v} - \frac{\mu}{\rho} (\vec{H} \cdot \nabla)\vec{H} + \nabla q = \vec{F}(x, t),
$$

$$\text{div} \vec{v} = 0, \quad \vec{H}_t = -\frac{1}{\mu} \text{rot} \vec{E}, \quad \text{div} \vec{H} = 0,$$

$$\text{rot} \vec{H} + \mu \sigma \beta \text{rot} \vec{H} \times \vec{H} + \mu \sigma \beta_1 \vec{H} \times (\text{rot} \vec{H} \times \vec{H}) = \sigma (\vec{E} + \mu \vec{v} \times \vec{H}).$$

(1.1)

Here $\vec{v}$ is the velocity vector field, $\vec{H}$ and $\vec{E}$ are the magnetic and electrical fields, $q = \frac{\mathcal{E}}{\rho} + \frac{\mu |\vec{H}|^2}{2\rho}$, $\rho$ is the pressure, $\vec{F}$ is the vector field of external forces, and $\rho, \nu, \mu, \sigma$ are constant coefficients (density, viscosity, magnetic permeability and electrical conductivity of the liquid). Finally, $\beta_1 = \text{const} \geq 0$ and $\beta = \text{const} \in \mathbb{R}$ (the Hall constant $\beta$ can be of arbitrary sign [1]). As usual, we have neglected the displacement current. Excluding $\vec{E}$, we arrive at the system

$$\vec{v}_t - \nu \Delta \vec{v} + (\vec{v} \cdot \nabla)\vec{v} - \frac{\mu}{\rho} (\vec{H} \cdot \nabla)\vec{H} + \nabla q = \vec{F}(x, t),$$

$$\vec{H}_t - \frac{1}{\mu \sigma} \Delta \vec{H} - \text{rot} [\vec{v} \times \vec{H}] - \beta \text{rot} (\vec{H} \times \text{rot} \vec{H}) -$$

$$-\beta_1 \text{rot} [(\vec{H} \times (\vec{H} \times \text{rot} \vec{H})] = 0, \quad \text{div} \vec{v} = 0, \quad \text{div} \vec{H} = 0,$$

(1.2)

which should be completed by initial and boundary conditions. We assume that the liquid is moving in a bounded container whose walls are made of perfectly conducting material, hence, $\vec{H} \cdot \vec{n} = 0$ and $\vec{E}_r = \vec{E} - \vec{n} (\vec{n} \cdot \vec{E}) = 0$ on the boundary. The vector field $\vec{E}$ can be found from the last equation in (1.1), so that the initial and boundary conditions take the form

$$\vec{v}(x, 0) = \vec{v}^{(0)}(x), \quad \vec{H}(x, 0) = \vec{H}^{(0)}(x),$$

$$\vec{v} \big|_{\partial \Omega} = \vec{A}, \quad \vec{H} \cdot \vec{n} \big|_{\partial \Omega} = 0,$$

$$\left\{ \frac{1}{\mu \sigma} \text{rot} \vec{H} - \beta [\vec{H} \times \text{rot} \vec{H}] - \beta_1 ([\vec{H} \times (\vec{H} \times \text{rot} \vec{H})] \right\} \bigg|_{\partial \Omega} = 0.$$

(1.3)

We restrict ourselves with consideration only of this initial boundary-value problem, although the boundary conditions
\[ \vec{v}|_{\partial \Omega} = \vec{a}, \quad \vec{H}|_{\partial \Omega} = 0, \]
and
\[ \vec{v} \cdot \vec{n}|_{\partial \Omega} = 0, \quad (S(\vec{v})\vec{n})|_{\partial \Omega} = 0, \quad \vec{H}|_{\partial \Omega} = 0, \]
with \( S_{ij} = \frac{\partial n_i}{\partial x_j} + \frac{\partial n_j}{\partial x_i} \), also have a certain physical sense.

In the case \( \beta_1 = \beta = 0 \), problem (1.2), (1.3) was studied in [2, 3]. In [3], a justification of the linearization principle was given not only for the equations of magnetohydrodynamics but also for more general equations of the form \( \frac{d}{dt} + A(t)u + K\vec{u} = f \). Here \( A(t) \) is a linear operator with a positive self-adjoint principal part and \( K \) is a nonlinear operator subordinate to \( A(t) \) in a certain sense. System (1.2) cannot be imbedded into this general framework due to the high order of nonlinear terms that describe the Hall and ion-slip currents. The questions of uniqueness and of stability of solutions of some initial boundary-value problems for Eqs. (1.2) are studied in [4–10], but the existence of a solution of problem (1.2), (1.3), close to a stationary solution, is established only in [8, 9] in the case \( \beta_1 = 0 \) (in [9] it is also required at the investigation of linear and nonlinear evolution problems that the Hall constant \( \beta \) is positive and sufficiently small). The solution is found in the space \( W^{4,2}_2(Q_T) \) which imposes additional restrictions on the compatibility of initial and boundary conditions; there appear compatibility conditions that contain the time derivative of the solution (they are not written explicitly). We mention also [11], where the displacement current is taken into account.

In the present paper, which is a continuation of [12], we also search for a solution of problem (1.2), (1.3) in the form \( \vec{H}(x,t) = \vec{H}_0(x,t) + \vec{h}(x,t), \quad \vec{v}(x,t) = \vec{V}_0(x,t) + \vec{u}(x,t) \), where \( \vec{H}_0 \) and \( \vec{V}_0 \) are certain given divergence-free vector fields. Clearly, \( \vec{u} \) and \( \vec{v} \) should be solutions to the problem
\begin{align*}
\vec{u}_t + A_1(\vec{u}, \vec{q}, \vec{h}) + K_1(\vec{u}, \vec{h}) = \vec{f}, \quad \text{div} \vec{u} = 0, \\
\vec{h}_t + A_2(\vec{h}, \vec{u}) + K_2(\vec{h}, \vec{u}) = \vec{g}, \quad \text{div} \vec{h} = 0, \\
\vec{u}|_{t=0} = \vec{u}_0, \quad \vec{h}|_{t=0} = \vec{h}_0, \quad \vec{u}|_{\partial \Omega} = \vec{a}, \\
\vec{h} \cdot \vec{n}|_{\partial \Omega} = 0, \quad B_\tau(\vec{h}) + K_\tau(\vec{h})|_{\partial \Omega} = \vec{d},
\end{align*}
where
\begin{align*}
A_1(\vec{u}, \vec{q}, \vec{h}) &= -\nu \Delta \vec{u} + (\vec{V}_0 \cdot \nabla) \vec{u} + (\vec{u} \cdot \nabla) \vec{V}_0 - \frac{\mu}{\rho} ((\vec{H}_0 \cdot \nabla) \vec{h} + (\vec{h} \cdot \nabla) \vec{H}_0) + \frac{1}{\rho} \nabla \vec{q}, \\
A_2(\vec{u}, \vec{h}) &= -\frac{1}{\sigma \mu} \text{grad div} \vec{h} + \text{rot} \vec{B}(\vec{h}) - \text{rot} [(\vec{V}_0 \times \vec{h}) + (\vec{u} \times \vec{H}_0)], \\
B(\vec{h}) &= \frac{1}{\sigma \mu} \text{rot} \vec{h} - \beta(\vec{H}_0 \times \text{rot} \vec{h} + \vec{h} \times \text{rot} \vec{H}_0) - \\
&- \beta_1 [\vec{H}_0 \times (\vec{H}_0 \times \text{rot} \vec{h}) + \vec{H}_0 \times (\vec{h} \times \text{rot} \vec{H}_0) + \vec{h} \times (\vec{H}_0 \times \text{rot} \vec{H}_0)], \\
K_1(\vec{u}, \vec{h}) &= (\vec{u} \cdot \nabla) \vec{u} - \frac{\mu}{\rho} (\vec{h} \cdot \nabla) \vec{h}, \\
K_2(\vec{h}, \vec{u}) &= \text{rot} K(\vec{h}) - \text{rot}(\vec{u} \times \vec{h}), \\
K(\vec{h}) &= -\beta(\vec{h} \times \text{rot} \vec{h}) - \beta_1 [(\vec{H}_0 \times (\vec{h} \times \text{rot} \vec{h}) + \vec{h} \times (\vec{H}_0 \times \text{rot} \vec{H}_0)] - \beta_1 (\vec{h} \times (\vec{h} \times \text{rot} \vec{h})).
\end{align*}
Finally, \( B_\tau(\vec{h}) \) and \( K_\tau(\vec{h}) \) in the boundary conditions are the tangential components of \( B(\vec{h}) \) and \( K(\vec{h}) \), respectively.

The main result of the paper is a theorem on the solvability of problem (1.4) in the anisotropic Sobolev spaces \( W^{2,1}_p(Q_T) \) \((Q_T = \Omega \times (0,T), p > 5/2)\) with the norm
\[ \|u\|_{W^{2,1}_p(Q_T)} = \left( \int_0^T \|u_t\|_{L^p_1(\Omega)}^p + \|u\|_{W^2_2(\Omega)}^p \right)^{1/p}. \]