VARIATIONAL PROBLEM OF THE TWO-PHASE MEDIUM ELASTICITY THEORY FOR THE ZERO SURFACE TENSION COEFFICIENT

V. G. Osmolovskii

Dedicated to N. N. Uraltseva on her jubilee

In this article, we give a proof of the existence theorem for an equilibrium state for the surface tension coefficient $\sigma = 0$ and investigate the behavior of the equilibrium state for small $\sigma$. Bibliography: 4 titles.

1. INTRODUCTION

In this paper, we continue to study the variational problem of phase transitions in solid medium mechanics (see [1-4]). Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary. We fix a number $p \in (1, \infty)$, a nonnegative function $\alpha \in L^1(\Omega)$, and a function $\beta(t), t \in \mathbb{R}_+$, with the following properties:

$$
\begin{align*}
\beta(t) &= t^r \quad \text{for} \quad m \geq p, \quad r \in [1, mp/(m-p)) \quad \text{for} \quad m > p, \quad r \in [1, \infty) \quad \text{for} \quad m = p; \\
\beta(t) &= \quad \text{is a continuous monotonically increasing nonnegative function.}
\end{align*}
$$

Let $M \in \mathbb{R}^{m \times m}$ be an $m \times m$ matrix, let $u \in \mathbb{R}^m$, and let $x \in \Omega$. We denote by $|\cdot|$ both the absolute value of a vector or scalar and the norm of a matrix.

We fix deformation energy densities $F^\pm(M, u, x)$ that satisfy the conditions

(a) the functions $F^\pm(M, u, x)$ are continuous in $M \in \mathbb{R}^{m \times m}$, $u \in \mathbb{R}^m$ for a.e. $x \in \Omega$, measurable in $x$ for all $M \in \mathbb{R}^m$, and satisfy the inequality

$$
|F^\pm(M, u, x)| \leq C[|M|^p + \beta(|u|)] + \alpha(x);
$$

(b) the derivatives $F^\pm_M(M, u, x)$ are continuous in $M \in \mathbb{R}^m$ for a.e. $x \in \Omega$, measurable in $x$ for all $M \in \mathbb{R}^m$, and satisfy the estimate

$$
|F^\pm_M(M, u, x)| \leq C[(|M|^{p-1} + \beta(|u|)^{1/p'}) + \alpha(x)^{1/p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1;
$$

(c) the functions $F^\pm(M, u, x)$ are convex with respect to the components of the matrix $M$ for all $u \in \mathbb{R}^m$ and for a.e. $x \in \Omega$;

(d) for all $M \in \mathbb{R}^m$ and for a.e. $x \in \Omega$, the coercivity condition

$$
C^{-1}[|M|^p - \lambda|u|^p] - \alpha(x) \leq F^\pm(M, u, x), \quad 0 < \lambda < \lambda^{-1}(\Omega)
$$

holds, where $\lambda(\Omega)$ is the constant in the Friedrichs inequality

$$
\int_{\Omega} |u|^p \, dx \leq \lambda(\Omega) \int_{\Omega} |\dot{u}|^p \, dx, \quad u \in \dot{W}_p^1(\Omega, \mathbb{R}^m),
$$

and $\dot{u}(x)$ is the matrix with components $\dot{u}_{ij} = \frac{\partial u_i}{\partial x_j}$.

Define the deformation energy functional $I_\sigma[u, \chi]$ on the set

$$u \in \hat{W}_p^1(\Omega, R^m), \quad \chi \in BV(\Omega), \chi(x) \text{ is a characteristic function},$$  

by the equality

$$I_\sigma[u, \chi] = \int_\Omega [\chi F^+(\dot{u}, u, x) + (1 - \chi)F^-(\dot{u}, u, x)] \, dx + \sigma \int_\Omega |D\chi|,$$  

where $\sigma > 0$ is the surface tension coefficient and

$$\int_\Omega |D\chi| = \sup_{h \in C_0^\infty(\Omega, R^m), |h| \leq 1} \int_\Omega \chi \, \text{div} \, h \, dx$$

is the variation of the function $\chi$. We say that a pair $\hat{u}, \hat{\chi}$ from the set (1.1) is an equilibrium state if it gives the global minimum to functional (1.2).

In [2], under conditions (a)-(d), an existence theorem for equilibrium states was established and some their properties were investigated. In this paper, we study the variational problem for the functional

$$I_0[u, \chi] = \int_\Omega [\chi F^+(\dot{u}, u, x) + (1 - \chi)F^-(\dot{u}, u, x)] \, dx$$

on the set of functions

$$u \in \hat{W}_p^1(\Omega, R^m), \quad \chi(x) \text{ is a measurable characteristic function}.$$  

We say that a pair of functions $\hat{u}, \hat{\chi}$ from the set (1.4) is an equilibrium state if it gives the global minimum to functional (1.3) on (1.4). We call this equilibrium state one-phase if one of the equalities $\hat{\chi}(x) = 0$ or $\hat{\chi}(x) = 1$ holds on a set of full measure. Otherwise, the equilibrium state is called two-phase.

It turns out that conditions (a)-(d) do not guarantee the existence of a global minimum of functional (1.3) on (1.4). We formulate an additional condition that provides the existence of a global minimum in terms of the function

$$\Phi(M, u, x) = F^-(M, u, x) - F^+(M, u, x).$$  

The structure of the paper is as follows. In Sec. 2, we study various modifications of the variational problem for functional (1.3) and their connections. In Sec. 3, under some restrictions on the function $\Phi$, we prove a theorem on the existence of an equilibrium state for the functional $I_0$. In Sec. 4, necessary conditions are given for the existence of one-phase equilibrium states for the functional $I_0$. Sufficient conditions are obtained for equilibrium states of this functional to be two-phase. We investigate the behavior of equilibrium states for functional (1.2) for small $\sigma$ in Sec. 5. Some results are stated that extend analogous results from [2, 4].

### 2. Various Modifications of the Variational Problem for the Functional $I_0$

Consider three sets of functions,

$$M_1 = \{\chi(x) : \chi \in BV(\Omega), \chi(x) \text{ is a characteristic function}\},$$

$$M_2 = \{\chi(x) : \chi(x) \text{ is a measurable characteristic function}\},$$

$$M_3 = \{\chi(x) : \chi(x) \text{ is measurable and } 0 \leq \chi(x) \leq 1 \text{ for a.e. } x \in \Omega\}.$$

Obviously, the inclusions $M_1 \subset M_2 \subset M_3$ hold. It is well known that the set $M_3$ is closed with respect to weak convergence in $L_q(\Omega)$, the closure of the sets $M_1$ and $M_2$ with respect to weak convergence in $L_q(\Omega)$ coincides with $M_3$, and for any function $\chi \in M_2$ there exists a sequence $\chi_n \in M_1$ such that $\chi_n(x) \to \chi(x)$ in $L_q(\Omega)$ as $n \to \infty$. In addition, if a sequence $\chi_n \in M_2$ converges weakly in $L_q$ to a function $\chi \in M_2$ for