THREE-DIMENSIONAL PROBLEMS OF STEADY VIBRATIONS OF ISOTROPIC PLATES

E. V. Altukhov, Yu. V. Mysovskii, and Yu. V. Panchenko

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Using the symbolic method of homogeneous solutions, we study problems of steady vibrations of isotropic plates. On the planar faces of the plate we state various kinds of homogeneous boundary conditions. We obtain the homogeneous solutions of the equations of motion and construct the dispersion equations. We carry out numerical analyses of the dispersion equations for a plate with clamped and planar faces. Four figures. Bibliography: 8 titles.

We consider an isotropic plate bounded by two parallel planes $\Gamma^+$ and $\Gamma^-$ at a distance $2h$ from each other and a series of cylindrical surfaces $\sigma_l$ $(l = 0, N)$ with generators perpendicular to the planes $\Gamma^+$ and $\Gamma^-$. The plate is deformed by external forces that vary harmonically in time and act on the surfaces $\sigma_l$. The stresses and displacements, or some combination of them are prescribed on the planar faces $\Gamma^+$ and $\Gamma^-$. Solving this problem reduces to integrating the equations of motion in displacements under prescribed boundary conditions. In the dimensionless coordinate system $x_i$ $(i = 1, 2, 3)$ we have [1; 2]:

\[
\Delta_2 \mathbf{U} + \nu_1 \lambda^2 \text{grad div} \mathbf{U} = 0; \\
\sigma_{i3}(x_1, x_2, \pm 1) = 0 \quad (i = 1, 2, 3); \\
U_i(x_1, x_2, \pm 1) = 0; \\
U_3(x_1, x_2, \pm 1) = 0, \quad \sigma_{j3}(x_1, x_2, \pm 1) = 0 \quad (j = 1, 2), \\
U_j(x_1, x_2, \pm 1) = 0, \quad \sigma_{33}(x_1, x_2, -1) = 0; \\
\sigma_{i3}(x_1, x_2, -1) = 0, \quad U_i(x_1, x_2, -1) = 0, \\
\sigma_{i3}(x_1, x_2, +1) = 0, \quad U_3(x_1, x_2, -1) = 0, \quad \sigma_{j3}(x_1, x_2, -1) = 0, \\
U_j(x_1, x_2, -1) = 0, \quad \sigma_{33}(x_1, x_2, +1) = 0, \quad \sigma_j(x_1, x_2, +1) = 0, \\
\sigma_{33}(x_1, x_2, -1) = 0, \quad \sigma_{i3}(x_1, x_2, -1) = 0; \\
\sigma_n|_{n=0} = f_1(s, x_3); \quad \sigma_{ns}|_{n=0} = f_2(s, x_3); \quad \sigma_{ns}|_{n=0} = f_3(s, x_3).
\]

Here

\[
\Delta_j = \Delta^2_j + \frac{\partial^2}{\partial x^2_3}; \quad \nabla^2_j = \nabla^2_0 + \omega_j^2; \quad \nabla^2 = \lambda^2 \nabla^2; \quad \omega_j^2 = \frac{\nu_1}{c_1^2 + \nu_2/c_2^2};
\]

$n$ and $s$ are the natural coordinates associated with the directrix of the cylindrical surface $\sigma_l$; $\omega$ is the frequency of vibrations; $c_1$ and $c_2$ are the velocities of transverse and longitudinal waves, respectively; and $f_i(s, x_3)$ are given functions. The rest of the notation is as in [2].

Using A. I. Lur'e's symbolic method [3] and applying the Laplace transform on the variable $x_3$, we obtain the general integral of the system (1):

\[
U_j(x_1, x_2, x_3) = C_2(x_3)U_{j0}(x_1, x_2) - \lambda^2 \omega_j^2 [C_1(x_3) - C_2(x_3)] \partial_2 \theta_0(x_1, x_2) + \\
+S_2(x_3)U'_{j0}(x_1, x_2) - \lambda^2 \omega_j^2 [S_1(x_3) - S_2(x_3)] \partial_1 \theta_0(x_1, x_2) \quad (j = 1, 2); \\
U_3(x_1, x_2, x_3) = S_2(x_3)U_{30}(x_1, x_2) + \lambda^2 \omega_1^2 [S_1(x_3) - S_2(x_3)] \nabla^2_0 \theta_0(x_1, x_2) + \\
+C_2(x_3)U'_{30}(x_1, x_2) - \lambda^2 \omega_2^2 [C_1(x_3) - C_2(x_3)] \theta_0(x_1, x_2),
\]


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where
\[ \theta_0(x_1, x_2) = \partial_1 U_{10}(x_1, x_2) + \partial_2 U_{20}(x_1, x_2) + \lambda^{-1} U_{30}'(x_1, x_2); \]
\[ \theta'_0(x_1, x_2) = \partial_1 U'_{10}(x_1, x_2) + \partial_2 U'_{20}(x_1, x_2) - \lambda^{-1} \nabla_2^2 U_{30}(x_1, x_2); \]
\[ C_j(x_3) = \cos \nabla_j x_3; \quad S_j(x_3) = (\sin \nabla_j x_3)/\nabla_j; \]
\[ U_{10}(x_1, x_2) \quad \text{and} \quad U'_{10}(x_1, x_2) \quad (i = 1, 2, 3) \] are the displacements and their derivatives with respect to \( x_3 \) in the middle plane. The functions \( U_{10}, U_{20}, \) and \( U_{30} \) characterize the stress state of the plate under expansion and contraction (Problem A) \[3\], \( U'_{10}, U'_{20}, \) and \( U'_{30} \) characterize it under bending (Problem B).

For conditions (2)–(4) Problems A and B can be studied separately, while in the case of condition (5) they can be studied together.

Substituting expressions (7) into the equations of Hooke’s law \[2\], and then substituting the resulting relations into Eq. (2), we shall have
\[ A \cdot X = 0. \] (8)

Here
\[ A = (a_{ij}) \quad (i, j = 1, 6), \quad X = (U_{10}, U_{20}, U_{30}, U'_{10}, U'_{20}, U'_{30}). \]
The elements \( a_{ij} \) of the matrix \( A \) are differential operators. The expressions for them are determined by the type of boundary conditions on the planar faces of the plate. In the case of boundary conditions (2) the expressions for \( a_{ij} \) are given in \[4\].

Using the method discussed in \[2\] to solve the system (8), we find that the components of the vector \( X \) can be expressed in terms of a function \( \psi(x_1, x_2) \) satisfying the operator dispersion equation
\[ (\det)\psi(x_1, x_2) = F(\nabla_0^2)\psi = V(\nabla_0^2)P(\nabla_0^2)\psi = 0, \] (9)

where \( F(\nabla_0^2) \) is an entire function of the two-dimensional Laplacian. In the case of conditions (2) the form of this function for Problems A and B is given in \[4\]. For conditions (3) and (4) respectively in Problem A we have:
\[ V(\nabla_0^2) = C_2, \quad P(\nabla_0^2) = \nabla_1^2 C_1 S_1 - \nabla_0^2 C_1 S_2; \] (10)
\[ V(\nabla_0^2) = \nabla_1^2 \nabla_2^2 S_2, \quad P(\nabla_0^2) = S_1 S_2, \quad V(\nabla_0^2) = C_2, \quad P(\nabla_0^2) = C_1 C_2. \] (11)

Similarly for Problem B we have
\[ V(\nabla_0^2) = S_2, \quad P(\nabla_0^2) = \nabla_2^2 C_1 S_1 - \nabla_0^2 C_2 S_1; \] (12)
\[ V(\nabla_0^2) = C_2, \quad P(\nabla_0^2) = \nabla_2^2 S_2, \quad \nabla(\nabla_0^2) = C_2 C_2, \quad P(\nabla_0^2) = S_1 S_2. \] (13)

Here \( C_i = C_i(1) \) and \( S_i = S_i(1) \). For the boundary conditions (5) we have
\[ V(\nabla_0^2) = C_{12}, \quad P(\nabla_0^2) = 4\nabla_0^2 \nabla_{02}^2 - (4\nabla_0^4 + \nabla_{02}^4)C_{12} S_{12} - (4\nabla_0^2 \nabla_1^2 \nabla_2^2 - \nabla_0^2 \nabla_{02}^2)S_{12} S_{22}; \]
\[ V(\nabla_0^2) = \nabla_2^2 S_{12}, \quad P(\nabla_0^2) = \nabla_2^2 S_{12} C_{12} - 4\nabla_0^2 \nabla_2^2 S_{12} C_{22}; \]
\[ V(\nabla_0^2) = C_{12}, \quad P(\nabla_0^2) = \nabla_1^2 S_{12} C_{12} - \nabla_0^2 S_{12} C_{12}; \]
\[ V(\nabla_0^2) = C_{12}, \quad P(\nabla_0^2) = \nabla_0^2 S_{12} C_{12} - \nabla_2^2 S_{12} C_{12}; \]
\[ V(\nabla_0^2) = C_{12} S_{22}, \quad P(\nabla_0^2) = 4\nabla_0^2 \nabla_{02}^2 - (4\nabla_0^4 + \nabla_{02}^4)S_{12} S_{22}; \]
\[ V(\nabla_0^2) = \nabla_2^2 S_{22}, \quad P(\nabla_0^2) = \nabla_2^2 S_{22} C_{12} - 4\nabla_0^2 \nabla_2^2 S_{22} C_{22}; \]
\[ V(\nabla_0^2) = C_{12}, \quad P(\nabla_0^2) = \nabla_1^2 S_{22} C_{12} - \nabla_0^2 S_{22} C_{12}; \]
\[ V(\nabla_0^2) = C_{12}, \quad P(\nabla_0^2) = \nabla_2^2 S_{22} C_{12} - \nabla_0^2 S_{22} C_{12}; \]
\[ V(\nabla_0^2) = C_{12} \tilde{N}_{22}, \]

\[ C_{12} = \cos 2\nabla_i, \quad S_{12} = (\sin 2\nabla_i)/\nabla_i \quad (i = 1, 2), \quad \nabla_{02}^2 = \nabla_0^2 + \nabla_2^2. \]

In accordance with Boggio’s theorem \[1\], we shall seek a solution of Eq. (9) in the set of metaharmonic functions
\[ \psi(x_1, x_2) = \sum_{(m)} \varphi_m(x_1, x_2), \quad \nabla_0^2 \varphi_m(x_1, x_2) = \gamma_m \varphi_m(x_1, x_2). \] (15)