WAVE SOLUTIONS OF EQUATIONS OF TIMOSHENKO TYPE FOR THE TRANSVERSE VIBRATIONS OF PLATES WITH PARAMETERS PERIODIC ON ONE COORDINATE

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The hyperbolic system of equations that describes the vibrations of plates inhomogeneous along one rectangular coordinate in the context of the Timoshenko theory is presented in canonical hamiltonian form, assuming the solution is periodic on a second coordinate. In the case of periodic inhomogeneity we study the structure of the solutions of certain wave boundary-value problems for plates of this type using the general properties of periodic hamiltonian systems. Bibliography: 6 titles.

In studying wave processes in deformable bodies with degenerate dimensions (rods, plates, and shells) using the applied theories the preservation of the hyperbolic properties of the three-dimensional Lamé equations [1] by the systems of equations of the approximate theories is of great importance. Equations of Timoshenko type satisfy this condition in the theory of transverse vibrations of thin plates. In the present article a system of equations of Timoshenko type is presented in canonical hamiltonian form under suitably chosen canonical variables for plates that are inhomogeneous on one coordinate and in the case when the solution depends harmonically on another coordinate and time. For periodically varying parameters we study the structure of solutions of the wave boundary-value problem on the basis of the properties of periodic hamiltonian systems.

The complete system of equations of Timoshenko type for transverse vibrations of isotropic plates consists of the following relations [1]:

\[
\begin{align*}
\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{21}}{\partial x_2} - Q_1 &= -D \frac{\partial^2 \varphi_1}{\partial x_1^2}, \\
\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} &= m \frac{\partial^2 w}{\partial t^2}, \\
M_{11} &= -D \left( \frac{\partial \varphi_1}{\partial x_1} + \nu \frac{\partial \varphi_2}{\partial x_2} \right), \\
M_{12} &= -\frac{1 - \nu}{2} D \left( \frac{\partial \varphi_1}{\partial x_2} + \nu \frac{\partial \varphi_2}{\partial x_1} \right), \\
Q_1 &= B_3 \left( \frac{\partial w}{\partial x_1} - \varphi_1 \right), \\
Q_2 &= B_3 \left( \frac{\partial w}{\partial x_2} - \varphi_2 \right).
\end{align*}
\]

Here \( \varphi_1, \varphi_2, \) and \( w \) are respectively the shear and deflection functions; \( M_{11} \) and \( M_{22} \) are the bending moments; \( M_{12} \) and \( M_{21} \) are the torques; \( Q_1 \) and \( Q_2 \) are the shear forces; \( D \) and \( B_3 \) are respectively the bending and shear stiffness; \( \nu \) is the Poisson coefficient; \( m \) and \( I \) are respectively the specific mass (per unit area) and moment of inertia (per unit of length). Here the quantities \( D, B_3, \nu, m, \) and \( I \) depend on the coordinate \( x_1 \).

We shall seek a solution of the system (1) in the general case in the form

\[
\begin{align*}
w(x_1, x_2, t) &= h_{00} \text{Re} q_1(\xi) \exp i(s x_2 - \omega t); \\
M_{11}(x_1, x_2, t) &= h_{00}^2 \text{Re} q_2(\xi) \exp i(s x_2 - \omega t); \\
\varphi_2(x_1, x_2, t) &= \text{Re} i q_3(\xi) \exp i(s x_2 - \omega t); \\
Q_1(x_1, x_2, t) &= h_{00} \text{Re} p_1(\xi) \exp i(s x_2 - \omega t); \\
\varphi_1(x_1, x_2, t) &= \text{Re} p_2(\xi) \exp i(s x_2 - \omega t); \\
M_{12}(x_1, x_2, t) &= -h_{00}^2 \text{Re} i p_3(\xi) \exp i(s x_2 - \omega t); \\
Q_2 &= B_3 \left( \frac{\partial w}{\partial x_2} - \varphi_2 \right).
\end{align*}
\]

where $\omega$ is the cyclic frequency; $s$ is the constant of propagation; $\xi = 2\pi x_1/x_{00}$ is a dimensionless coordinate, $\rho_{00}$ and $x_{00}$ are normalizing parameters having dimension of length; and $\rho_{00}$ is a normalizing parameter having the dimension of the Young's modulus. Then to determine the dimensionless functions $q_i(\xi)$ and $p_i(\xi)$ we obtain from Eqs. (1)

\[
2\pi\varepsilon\frac{dq_1}{d\xi} = p_1/\bar{B}_3(\xi) + p_2, \\
2\pi\varepsilon\frac{dq_2}{d\xi} = p_1 + \bar{l}(\xi)\bar{\omega}^2p_2 - \bar{s}p_3, \\
2\pi\varepsilon\frac{dq_3}{d\xi} = -\bar{s}p_2 + 2(1 - \nu(\xi))p_3/\bar{D}(\xi), \\
2\pi\varepsilon\frac{dp_1}{d\xi} = (\bar{B}_3(\xi)\bar{s}^2 - \bar{m}(\xi)\bar{\omega}^2)q_1 - \bar{B}_3\nu(\xi)\bar{s}q_3, \\
2\pi\varepsilon\frac{dp_2}{d\xi} = -q_2/\bar{D}(\xi) + \nu(\xi)\bar{s}q_3, \\
2\pi\varepsilon\frac{dp_3}{d\xi} = -\bar{B}_3(\xi)\bar{s}q_1 + \nu(\xi)\bar{s}q_2 + (\bar{a}(\xi) - \bar{l}(\xi)\bar{\omega}^2)q_3.
\]

Here

\[
\bar{D} = D/h_{00}^3, \quad \bar{B}_3 = B_3/h_{00}c_{00}, \quad \bar{m} = m/h_{00}\rho_{00}, \quad \bar{l} = I/\rho_{00}h_{00}^3, \\
\bar{s} = s_{00}, \quad \bar{\omega} = \omega h_{00}\sqrt{\rho_{00}/\rho_{00}}, \quad \bar{a}(\xi) = \bar{B}_3(\xi) + \bar{l}(\xi)(1 - \nu^2(\xi))\bar{s}^2.
\]

In the special case of a plate of width $a_2$ we represent the solutions of the system (1) satisfying the conditions for contacting edges $x_2 = 0$ and $x_2 = a_2$ by trigonometric series

\[
w(x_1, x_2, t) = h_{00}\text{Re}e^{-i\omega t}\sum_{n=1}^{\infty} q_{1,n}(\xi) \sin s_n x_2, \\
M_{11}(x_1, x_2, t) = h_{00}^2\text{Re}e^{-i\omega t}\sum_{n=1}^{\infty} q_{2,n}(\xi) \sin s_n x_2, \\
\phi_2(x_1, x_2, t) = \text{Re}e^{-i\omega t}\sum_{n=1}^{\infty} q_{3,n}(\xi) \cos s_n x_2, \\
Q_1(x_1, x_2, t) = h_{00}\text{Re}e^{-i\omega t}\sum_{n=1}^{\infty} p_{1,n}(\xi) \sin s_n x_2, \\
\phi_1(x_1, x_2, t) = \text{Re}e^{-i\omega t}\sum_{n=1}^{\infty} p_{2,n}(\xi) \sin s_n x_2 \\
M_{12}(x_1, x_2, t) = -h_{00}^2\text{Re}e^{-i\omega t}\sum_{n=1}^{\infty} p_{3,n}(\xi) \cos s_n x_2.
\]

Here the functions $q_{i,n}(\xi)$ and $p_{i,n}(\xi)$ are also determined from a system of the form (3), in which it is necessary to replace $s$ by $s_n = n\pi/a_2$.

We now introduce the $3 \times 3$ symmetric matrices

\[
Q(\xi) = \frac{1}{2\pi\varepsilon} \begin{bmatrix}
\frac{1}{\bar{B}_3(\xi)} & 1 & 0 \\
1 & \bar{l}(\xi)\bar{\omega}^2 & -\bar{s} \\
0 & -\bar{s} & 2(1 - \nu(\xi))/\bar{D}(\xi)
\end{bmatrix}, \\
P(\xi) = \frac{1}{2\pi\varepsilon} \begin{bmatrix}
\bar{m}(\xi)\bar{\omega}^2 - \bar{B}_3(\xi)\bar{s}^2 & 0 & \bar{B}_3(\xi)\bar{s} \\
0 & 1/\bar{D}(\xi) & -\nu(\xi)\bar{s} \\
\bar{B}_3(\xi)\bar{s} & -\nu(\xi)\bar{s} & \bar{l}(\xi)\bar{\omega}^2 - \bar{a}(\xi)
\end{bmatrix}
\]