DESCRIPTION OF THE SET OF STRONG LIMITS OF NONREFLECTIVE JACOBIAN MATRICES

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1. Let $J$ denote a tridiagonal matrix that is infinite in both directions and has coefficients

$$J_{ij} = b(j)\delta(i - j - 1) + a(i)\delta(i - j) + b(i)\delta(i - j + 1)$$

$(i, j \in \mathbb{Z}, \text{ Im } a(i) = 0, \ b(i) > 0)$,

and let $P(\lambda, n)$ and $Q(\lambda, n)$ constitute a fundamental system of solutions for the finite-difference equation

$$b(n - 1)\psi(n - 1) + a(n)\psi(n) + b(n)\psi(n + 1) = \lambda\psi(n)$$

(1.1)

with initial data

$$P(-1) = 0, \ Q(-1) = 1,$$

$$P(0) = 1, \ Q(0) = 0.$$

According to the Weyl–Hellinger theorem [1], there exist functions $m^\pm(\lambda)$ that are holomorphic on the entire real axis and such that the solutions

$$\psi^+(\lambda, n) = m^+(\lambda)P(\lambda, n) + Q(\lambda, n),$$

$$\psi^-(\lambda, n) = P(\lambda, n) + m^-(\lambda)Q(\lambda, n)$$

of Eq. (1.1) belong to the space $l_2(\mathbb{Z}_\pm)$ and

$$-b(-1)\frac{\text{Im } m^+(\lambda)}{\text{Im } \lambda} = \sum_{k=0}^{\infty} |\psi^+(\lambda, k)|^2, \quad -b(-1)\frac{\text{Im } m^-(\lambda)}{\text{Im } \lambda} = \sum_{k=-\infty}^{-1} |\psi^-(\lambda, k)|^2.$$

The functions $m^\pm(\lambda)$ take conjugate values at conjugate points and

$$\frac{\text{Im } m^\pm(\lambda)}{\text{Im } \lambda} > 0, \quad m^\pm(\lambda) = \frac{b(-1)}{\lambda} + O(\lambda^{-2}), \quad \text{Im } \lambda \to \infty.$$

Following [2], we replace the pair of functions $m^\pm(\lambda)$ with the one function $m(z)$ defined by the equations

$$m(z) = \begin{cases} m^+(z + z^{-1}), & |z| < 1 \\ (m^-(z + z^{-1}))^{-1}, & |z| > 1. \end{cases}$$

and we call it the Weyl function of the matrix $J$. The function $m(z)$ uniquely determines the entire matrix $J$ and has the following properties:

1) $m(z)$ is holomorphic off the real axis and the unit circle:

2) $\frac{\text{Im } m(z)}{\text{Im } z} > 0 (|z| \neq 1, \ \text{Im } z \neq 0)$;

3) $\lim_{t \to +\infty} \frac{t}{m(it)} = \lim_{t \to +\infty} \frac{m(it)}{it} = b(-1)$; (see [2], Lemma 1.1).

A matrix is said to be rapidly decreasing if

$$\sum_{n=-\infty}^{\infty} n|a(n)| < \infty, \quad \sum_{n=-\infty}^{\infty} n|b^2(n) - 1| < \infty.$$
For rapidly decreasing Jacobian matrices we can replace the standard spectral characteristics (Weyl functions and spectral functions) by introducing alternative characteristics that appear in scattering theory (reflection coefficients, discrete spectra, and normalizing coefficients). Rapidly decreasing matrices for which the reflection coefficient is zero are said to be nonreflective. It was shown in [2] that a function \( m(z) \) is the Weyl function of a nonreflective matrix if and only if it admits a representation of the form

\[
m(z) = C \left( z + \sum_{i=1}^{N} \frac{-\alpha_j}{\lambda_j - z} - \sum_{j=1}^{N} \frac{-\alpha_j}{\lambda_j} \right),
\]

where \(-\alpha_j > 0\), and the \( \lambda_j \in \mathbb{R} \) are pairwise different numbers, and if \( ||J|| < M + M^{-1} \), then

\[
M^{-1} < |\lambda_j| < M
\]

(see [2], (2.17), Lemma 2.2 and Theorem 2.2). Here \( C = b^{-1}(-1) \) and it follows from the formula

\[
\lim_{z \to \infty} \frac{m(z)}{z} = b(-1)
\]

that \( b^2(-1) - 1 + \sum_{j=1}^{N} \frac{-\alpha_j^2}{\lambda_j^2} \). Let \( \rho(\lambda) \) denote the discrete measure

\[
d\rho(\lambda) = \sum_{j=1}^{N} \alpha_j \delta(\lambda - \lambda_j) d\lambda.
\]

Then representation (1.2) can be written in the form

\[
m(z) = C \left( z + \int \frac{d\rho(\lambda)}{\lambda - z} - \int \frac{d\rho(\lambda)}{\lambda} \right);
\]

\[
C = b^{-1}(-1) = \left( 1 + \int \frac{d\rho(\lambda)}{\lambda^2} \right)^{-1/2}.
\]

The fundamental result of the present paper is the following theorem.

**Theorem.** A function \( m(z) \) is the Weyl function of the strong limit of a sequence of nonreflective Jacobian matrices if and only if it admits a representation of the form (1.4), where \( \rho(\lambda) \) is a Borel measure that is concentrated on the real axis and does not contain zero.

In order to prove our theorem we first make the following remark. Let \( \{J_N\} \) be a sequence of Jacobian matrices, and let \( m_N(z) \) be their Weyl functions. Let \( J_N \to J \) in the sense of strong convergence, and let \( m_N(z) \to m(z) \) for each \( z \) (\( |z| \neq 1, \text{Im} z \neq 0 \)). Then \( m(z) \) is the Weyl function of the matrix \( J \). Indeed, the canonical solutions \( P_N(\lambda, n), Q_N(\lambda, n) \) corresponding to the matrices \( J_N \) for each \( \lambda \) converge to the canonical solutions \( P(\lambda, n), Q(\lambda, n) \) of the equation \( J\psi = \lambda \psi \). When \( \lambda = z + z^{-1} \) (\( |z| \neq 1, \text{Im} z \neq 0 \)), the functions

\[
\psi_N(z, n) = m_N(z)P_N(z + z^{-1}, n) + Q_N(z + z^{-1}, n),
\]

are the Weyl solutions satisfying the equations \( J_N\psi_N(z, n) = (z + z^{-1})\psi_N(z, n) \). In addition, when \( |z| < 1 \)

\[
-b_N(-1) \frac{\text{Im} m_N(z)}{\text{Im}(z + z^{-1})} = ||\psi_N(z, n)||_{L^2(z_+)} ,
\]

and when \( |z| > 1 \),

\[
-b_N(-1) \frac{\text{Im} m_N(z)}{\text{Im}(z + z^{-1})} = ||\psi_N(z, n)||_{L^2(z_-)} .
\]

When we pass to the limit in Eqs. (1.5) and (1.6) as \( N \to \infty, \) we find that \( m(z) = \lim m_N(z) \) is the Weyl function for the matrix \( J \).

We now turn to proof of the theorem.

**Necessity.** Let \( \{J_N\}_{N=1}^{\infty} \) be a sequence of nonreflective Jacobian matrices, and let \( J_N \to J \) in the sense of strong convergence. The nonreflective matrices \( J_N \) are bounded, and since they have strong limits, they are also bounded taken together. Let \( ||J_N|| < M + M^{-1} \forall N \in \mathbb{N} \). The Weyl functions \( m_N(z) \) for the matrices \( J_N \) are of the form

\[
m_N(z) = C_N \left( z + \int \frac{d\rho_N(\lambda)}{\lambda - z} - \int \frac{d\rho_N(\lambda)}{\lambda^2} \right),
\]

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