TAME REPRESENTATIONS OF THE HECKE ALGEBRA $H(\infty)$ AND THE $q$-ANALOGS OF PARTIAL BIJECTIONS

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In this paper the $q$-analogs of the semigroup of partial bijections are introduced and studied. They are applied to the description of the so-called tame representations of the infinite-dimensional Hecke algebra. Bibliography: 3 titles.

§1. THE MAIN DEFINITIONS

1.1. The finite-dimensional Hecke algebra $H(\infty)$ is a complex algebra with unity which depends on the parameter $q > 0$ and is given by the generators $\sigma_i, i = 1, 2, \ldots,$ and by the relations

\begin{align*}
&(\sigma_i - q)(\sigma_i + 1) = 0, \\
&\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i - j| > 1, \\
&\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.
\end{align*}

Consider the involution $\sigma_i^* = \sigma_i$ in $H(\infty)$. Only the $*$-representations of $H(\infty)$ in a Hilbert space will be of main interest. There are exactly two one-dimensional representations of $H(\infty)$:

\begin{align*}
T^+(\sigma_i) &= q, \quad i = 1, 2, \ldots, \\
T^-(\sigma_i) &= -1, \quad i = 1, 2, \ldots.
\end{align*}

The vectors that are transformed in accordance with the representations $T^+$ and $T^-$ will be called invariants and antiinvariants, respectively. The representations are transformed one into the other by the automorphism

$$\sigma_i \mapsto -\sigma_i + (q - 1)$$

of the algebra $H(\infty)$; therefore, the whole subsequent theory is developed for invariants and antiinvariants in parallel.

1.2. We denote by $H(k)$ and $H_k(\infty)$ the subalgebras in $H(\infty)$ generated by the elements $\sigma_1, \ldots, \sigma_{k-1}$ and $\sigma_{k+1}, \ldots,$ respectively. Obviously, these subalgebras commute. The subalgebra generated by the elements $\sigma_i, i \neq k$, will be identified with $H(k) \otimes H_k(\infty)$. Let $T$ be a representation of $H(\infty)$ in a Hilbert space $V$. We put

$$V_k = \{v \in V, T(\sigma_i)v = qv, i > k\},$$

and we define the subspace of antiinvariants $V_k^-$ in a similar way.

**Definition.** An irreducible $*$-representation of the algebra $H(\infty)$ in the Hilbert space $V$ is said to be tame, provided that $V_k \neq 0$ (or $V_k^- \neq 0$) for some $k$.

Following paper [1], we obtain a classification of tame representations of the algebra $H(\infty)$.

Obviously, the subspace $V_k$ is invariant relative to the subalgebra $H(k)$, and $V_k \subset V_{k+1}$. We put $V_\infty = \bigcup_k V_k$; the subspace $V_\infty$ is nonempty and invariant relative to the algebra $H(\infty)$. Since the representation is assumed to be irreducible, the subspace $V_\infty$ is dense in $V$.


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Example. Let $e_1, e_2, \ldots$ be an orthonormalized basis in the Hilbert space $V$. We put

$$T(\sigma_i) = \begin{cases} q^{1/2}e_{i+1}, & j = i, \\ (q - 1)e_{i+1} + q^{1/2}e_i, & j = i + 1, \\ qe_j, & j \neq i, i + 1. \end{cases}$$

It is easy to see that this is a $*$-representation of the algebra $H(\infty)$ (for $q = 1$ this is simply the action of the permutation group $S(\infty)$ by the permutations of the basis vectors). Clearly, for $q \geq 1$ the subspace $V_k$ is spanned by the vectors $e_1, \ldots, e_k$. We show that in this case the representation $T$ is irreducible. Let $W \subset V$ be an invariant subspace, and let $P$ be the projection onto it. Obviously, $Pe_1 \in V_1 = C e_1$. Therefore, either $Pe_1 = e_1$ and $e_1 \in W$ or $Pe_1 = 0$ and $e_1 \in W^\perp$. But the vector $e_1$ generates $V$ as an $H(\infty)$-module, and either $W = V$, or $W = 0$. Thus, in the case $0 < q < 1$, this representation is tame and irreducible.

In the case $0 < q < 1$, the vector

$$\sum_{i=1}^{\infty} q^{i/2}e_i,$$

which is invariant relative to the whole algebra $H(\infty)$, appears. In this case the representation $T$ is reducible as well as the corresponding representation of a finite-dimensional Hecke algebra. An irreducible tame representation of the algebra $H(\infty)$ is realized in the orthogonal complement of the invariant vector.

1.3. We see that even in this simplest example the cases $q > 1$ and $0 < q < 1$ are distinguished. However, this distinction is not essential because the mapping

$$\sigma_i \mapsto q^{-1}\sigma_i + q^{-1} - 1$$

is an isomorphism of the Hecke algebra corresponding to the parameter $q$ and the Hecke algebra corresponding to the parameter $q^{-1}$. However, this distinction will occur repeatedly and it is related to the following fact. Recall that the symbol $[k]$ defined by the formula

$$[k] = \frac{q^k - 1}{q - 1}$$

is usually called the $q$-analog of the number $k$. From the definition, it is obvious that

$$\frac{1}{[\infty]} = \begin{cases} 0, & q \geq 1 \\ 1 - q, & 0 < q \leq 1. \end{cases}$$

We observe that for $q = 1$ these two formulas coincide.

§2. The algebras $\Gamma_{<q}$ and $\Gamma_{q<}$

2.1. In the classification of tame representations of $S(\infty)$ obtained in [1], the main part was played by a semigroup of partial bijections $\Gamma(k)$ containing the permutation group $S(k)$. The semigroup $\Gamma(k)$ is defined as follows. A partial bijection $\gamma$ of the set with $k$ points is given by a pair of subsets

$$\text{dom}(\gamma), \text{im}(\gamma) \subset \{1, \ldots, k\}$$

and a bijection

$$\gamma : \text{dom}(\gamma) \rightarrow \text{im}(\gamma).$$

The composition of such partial bijections is defined in a natural way. The group of invertible elements of the semigroup $\Gamma(k)$ coincides with the group $S(k)$. It is convenient to assume that $\gamma(i)$ is defined everywhere and that

$$\gamma(i) = \begin{cases} \gamma(i), & i \in \text{dom}(\gamma) \\ \emptyset, & \text{otherwise}. \end{cases}$$

The semigroup $\Gamma(k)$ may be thought of as the semigroup of matrices composed of ones and zeros and such that in every row and in every column there is at most one 1.