SOME REMARKS ON HOMOCLINIC GROUPS OF HYPERBOLIC TORAL AUTOMORPHISMS

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The homoclinic group (an invariant with respect to topological conjugacy) for hyperbolic toral automorphisms is determined. Certain conditions are given for conjugacy of a homeomorphism of a compact space to hyperbolic toral automorphism. Bibliography: 7 titles.

Let $X$ be a metrizable compactum and $HOM(X)$ be the group of homeomorphisms of $X$ (upon itself). For $T \in HOM(X)$ the homoclinic equivalence relation $\sim_T$ is introduced in the following way: $x \sim_T y$ iff $d(T^n x, T^n y) \to 0$, where $d$ is some metric compatible with the topology on $X$.

The homoclinic group $HCL_T$ is defined by the formula

$$HCL_T = \{S \mid S \in HOM(X), \quad T^n ST^{-n} \to I\}.$$ 

In this definition, $I$ is the unit in the group $HOM(X)$, and the convergence is regarded in the sense of the metric

$$\rho(T_1, T_2) = \max_{x \in X} d(T_1 x, T_2 x) + \max_{x \in X} d(T_1^{-1} x, T_2^{-1} x).$$

The group $HOM(X)$ is complete in the metric $\rho$, and the group operations are continuous with respect to $\rho$. It is possible to exclude the metric $d$ from our consideration by defining the topology on $HOM(X)$ in terms of the uniformity on $X$ (uniquely determined by the structure of the compactum).

Next, this topology defines left and right uniformities on $HOM(X)$, while the weakest of the uniformities majorizing both of them is just the one defined by the metric $\rho$. Using uniformities instead of metrics allows us to relinquish the assumption that $X$ is metrizable.

It is obvious that the algebraic type of the group $HCL_T$ is an invariant of homeomorphisms of the compactum (onto itself) under topological conjugacy. Some other invariants of the homeomorphism $T$ can also be defined in terms of the relation $\sim_T$, of the group $HCL_T$ and its action on $X$, of the action of the homeomorphism $T$ on $HCL_T$ (by conjugation), and of the set of homoclinic equivalence classes (which is highly nonregular in interesting situations). In some cases such a homoclinic system of invariants (defined intrinsically in topological terms) allows us to characterize some class of homeomorphisms, and, with the help of additional invariants, individual homeomorphisms in this class. In this paper it is the class of hyperbolic toral automorphisms and of homeomorphisms topologically conjugated with them. (And, according to well-known results [4, 6], this class includes all Anosov toral diffeomorphisms).

The action of the group $HCL_T$ on $X$ induces a partition of $X$ into orbits. It is easy to see that this partition is finer than the partition into the homoclinic equivalence classes, i.e., $x \sim y$ if $y = Sx$ for some $S \in HCL_T$. The inverse statement is false, in general. It can be verified by considering hyperbolic automorphisms of compact nilmanifolds. In any case, the triviality of the relation $\sim_T$ (i.e., the situation when each point $x \in X$ is homoclinic only to itself) implies the triviality of $HCL_T$: in this case $HCL_T = \{I\}$. One of the earliest sources dealing with $\sim_T$ and related questions is the book [7] by D. Ruelle, where the reference to the original work by D. Capocaccia can be found (homoclinic points are called conjugated by these authors, and their main aim is elaborating the definition of the Gibbs measure which is independent of the choice of a symbolic representation). Similar notions were considered in [2, 3, 5] in other contexts.
Now, let $X = \mathbb{T}^d (= \mathbb{R}^d / \mathbb{Z}^d)$ be the $d$-dimensional torus, $T$ be an algebraic automorphism of the torus $\mathbb{T}^d$ (such an automorphism, as it is known, is defined, in the standard basis of the space $\mathbb{R}^d$, by a $d \times d$-matrix $M$ over integers with $\det(M) = \pm 1$). Having fixed an ergodic automorphism $T$ of the torus $\mathbb{T}^d$ (ergodicity is equivalent to the absence of roots of the unit in the spectrum of $M$), we shall study the relation $\sim_T$ and the group $HCL_T$. In this paper we shall restrict ourselves to the case of a hyperbolic automorphism of the torus (in terms of the matrix $M$, it means the absence of eigenvalues $\lambda$ with $|\lambda| = 1$). Although the objects we are interested in can be nontrivial for nonhyperbolic automorphisms as well, it is in the hyperbolic case that they can be described especially clearly.

First, we note that for a hyperbolic automorphism the equivalence relation $\sim_T$ can be described in the following way. Due to the existence on the torus $\mathbb{T}^d$ of a metric invariant under translations, and to the fact that the neutral element $0$ is a fixed point for $T$, it is clear that $x \sim_T y$ iff $x - y \in \Gamma$, where

$$\Gamma = \{ \gamma | \gamma \in \mathbb{T}^d, T^n \gamma \to \infty \}. $$

The group $\Gamma$ was considered in [5], where it was established that $\Gamma$ is a countable dense $T$-invariant subgroup of $\mathbb{T}^d$ which is a free Abelian group of rank $d$. Each element $\gamma \in \Gamma$ induces the translation $\gamma : \mathbb{T}^d \to \mathbb{T}^d$, $\gamma x = \gamma + x$, $x \in \mathbb{T}$, which is a homoclinic homeomorphism for $T$. Thus, $\Gamma$ is isomorphically and $T$-equivariantly embedded into $HCL_T$ as a subgroup of homoclinic translations $\Gamma_T$.

**Proposition 1.** $HCL_T = \Gamma_T$.

**Proof.** Let $S \in HCL_T$. Then, for each $x \in \mathbb{T}^d$, we have $Sx \sim_T x$ and, consequently, $Sx - x \in \Gamma \subset \mathbb{T}^d$. Thus, $S - I$ continuously maps $\mathbb{T}^d$ into a countable subset of $\Gamma$. Since $\mathbb{T}^d$ is connected, $S - I$ is constant, i.e., there exists $\gamma \in \Gamma$ such that $Sx = x + \gamma$ for every $x \in X$. This proves the proposition.

**Remark 1.** Since, according to A. Manning's theorem, any Anosov diffeomorphism of the torus $\mathbb{T}^d$ is conjugated to an algebraic one, Proposition 1 and the description of $\Gamma$ given in [5] imply that for any Anosov diffeomorphism of the torus $\mathbb{T}^d$ the homoclinic group is a free Abelian group on $d$ generators acting freely and transitively on homoclinic equivalence classes.

**Remark 2.** There is another explanation of the formula $HCL_T = \Gamma_T$ and of the coincidence of both groups with the fundamental group of the torus, which is more geometric. This explanation, which does not use the group structure of $\mathbb{T}^d$, is the following. With each pair of points $x, y \in \mathbb{T}^d$, $x \sim_T y$, we connect a loop obtained by the composition of any path leading from $x$ to $y$ along a stable leaf (in the considered case of an algebraic automorphism of the torus the mentioned leaf is simply a co-set modulo the stable subgroup of the point 0) and any path leading from $y$ to $x$ along the nonstable leaf (it is also a co-set modulo a subgroup). Since leaves are homeomorphic to Euclidian spaces (it is immediately seen in our case, but it is true for the general Anosov diffeomorphism as well see [4]), the homotopic type $l(x, y)$ of such a loop (with the beginning and the end at the point $x$) is uniquely defined. It can be shown, by consideration of the universal cover, that the thus defined mapping of the homoclinic equivalence class of the point $x \in X$ into the fundamental group $\pi_1(\mathbb{T}^d, x)$ is one-to-one (the analogous statement is also true for hyperbolic automorphisms of compact nilmanifolds; the author hopes to describe the homoclinic situation in this case in a separate publication).

In view of the above, the homoclinic equivalence class of the point $x$ is parametrized by the group $\pi_1(\mathbb{T}^d, x)$ in a natural way. For another point $y \in \mathbb{T}^d$ the corresponding class is parametrized by $\pi_1(\mathbb{T}^d, y)$. However, $\pi_1(T, x)$ and $\pi_1(T^d, y)$ are canonically isomorphic (here the commutativity of the fundamental group of the torus $\mathbb{T}^d$ plays its role: for the general arcwise connected space $X$, $\pi_1(X, x)$ and $\pi_1(X, y)$ are still isomorphic, but only up to an inner automorphism, thus not canonically). This allows us, having fixed an element of the fundamental group, to construct the corresponding homoclinic homeomorphism by associating with each point $x \in \mathbb{T}^d$ the point $y \in \mathbb{T}^d$, $x \sim_T y$, such that $l(x, y)$ coincides with the prescribed element of the fundamental group. Conversely, a homoclinic homeomorphism $S$ determines the element $l(x, Sx)$ of the fundamental group (independent of $x \in \mathbb{T}^d$).