SUBORDINATORS AND THE ACTIONS OF PERMUTATIONS WITH QUASI-ININVARIANT MEASURE

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We introduce a class of probability measures in the space of virtual permutations associated with subordinators (i.e., processes with stationary positive independent increments). We prove that these measures are quasi-invariant under both left and right actions of the countable symmetric group \( \mathfrak{S}_\infty \), and a simple formula for the corresponding cocycle is obtained. In case of a stable subordinator, we find the value of the spherical function of a constant vector on the class of transpositions. Bibliography: 19 titles.

INTRODUCTION

In paper [7], the authors introduced the space of virtual permutations \( \mathfrak{S}_\infty = \lim \mathfrak{S}_n \), endowed with two commuting actions (left and right) of the countable symmetric group \( \mathfrak{S}_\infty = \lim \mathfrak{S}_n \). A remarkable class of probability measures on the space \( \mathfrak{S}_\infty \), quasi-invariant for these actions, was considered.

Given an action of a group preserving the type of measure, one can construct, in a standard way, a unitary representation of the group in the space of square integrable functions. The unit constant determines a distinguished ("vacuum") vector in this space. The purpose of the paper [7] was to study the spherical function of the vacuum vector and to develop harmonic analysis for the corresponding unitary representations.

The measures \( \mu_\theta \) studied in [7] depend on a positive parameter \( \theta \) and can be constructed on the basis of a simple principle: the measure of a permutation \( w \in \mathfrak{S}_n \) is proportional to \( \theta^c \), where \( c = c(w) \) is the number of cycles of this permutation. In paper [3] these measures were referred to as Ewens measures. The theory of such measures is equivalent to the theory of Poisson–Dirichlet distributions (\( PD(\theta) \)-measures) in the space of series with positive monotonically decreasing terms and with unit sum (see [1, 10]). Another technically more convenient way to work with Ewens measures, is based on the notion of GEM-measures: in the space of discrete probability distributions arising in the simplest version of the stick breaking process.

In this note, we consider more general measures in the space \( \mathfrak{S}_\infty \) which are quasi-invariant under both left and right actions of the countable symmetric group \( \mathfrak{S}_\infty \). They are connected with subordinators, i.e., random processes with stationary independent positive increments. We essentially use well-developed probabilistic theory of such processes, see [12, 15, 16]. The main result (formula (3.4.1)) consists of the proof of the quasi-invariance of measures connected with subordinators and of the computation of the corresponding density (cocycle).

The measures on the space of virtual permutations connected with subordinators include Ewens measures; they correspond to the gamma-process. Another remarkable class of subordinators is that of stable subordinators, for which the distributions of increments are governed by stable laws. These processes arise in two different contexts: in the study of excursions of Brownian motion and the Bessel processes of small dimension, and also as limits of specific distributions on symmetric groups generalizing Ewens distributions.

In the case of stable subordinators, we compute the value of the spherical function corresponding to the constant vector on the class of transpositions (formula (4.4.1)). It would be interesting to obtain its values on other conjugacy classes, too.

The plan of the paper is as follows.

We begin in \( \S 0 \) with some general remarks about virtual permutations and the way they are determined by sequences \( x \in [0, 1]^\infty \).

In \( \S 1 \), we define the branching graph of compositions that covers the branching graph of conjugacy classes of symmetric groups, and show that the generalized stick breaking process (SBP), introduced in...
[3], determines central measures for this graph. We point out a somewhat unexpected appearance of the Fibonacci lattice in connection with random permutations.

In §2, we find the general requirements for the distribution of records of a sequence \( x \) which cause a random virtual permutation \( \omega(x) \in \mathfrak{S}_\infty \) to be invariant under “inner automorphisms” from the group \( \mathfrak{S}_\infty \).

The contents of Secs. 1 and 2 should be considered as a slightly different presentation of Pitman’s results [13] concerning partially exchangeable partitions.

As Kingman pointed out long ago in [9], not only the gamma-process, but other subordinators as well, generate interesting partition structures, or, in our context, virtual permutations. In §3, we specialize the generalized stick breaking process of [7] to obtain measures in the space \( \mathfrak{S}_\infty \) associated with subordinators. We call them the PPY-measures since the definition is based on the Perman–Pitman–Yor theorem from [12]. In Secs. 3.3–3.4 we prove the quasi-invariance of PPY-measures.

Key examples connected with the gamma-process and stable subordinators are considered in §4. Following [12], we introduce two-parameter generalizations of GEM-distributions and Poisson–Dirichlet distributions and compute the simplest non-trivial value of the spherical function on \( \mathfrak{S}_\infty \) associated with these families.

In §5, we describe the generalization of the Ewens formula due to Pitman and establish a relation of Ewens–Pitman distributions in the space of virtual permutations to GEM-distributions from §4. The results of the preprint [15] on asymptotic behavior of the cycle structure of a random permutation are used in alternative computation of cocycles of quasi-invariant measures associated with stable subordinators.

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§0. REAL SEQUENCES AND VIRTUAL PERMUTATIONS

We start with preliminary remarks on virtual permutations and their parameterization by points of the cube \([0,1]^\infty\).

0.1. Virtual permutations. We recall (see [7]) that the induced permutation \( \tau = \sigma' \in \mathfrak{S}_{n-1} \) for \( \sigma \in \mathfrak{S}_n \) is obtained by removing the largest element \( n \) from the corresponding cycle of the permutation \( \sigma \). A virtual permutation is a sequence \( w = (w_1, w_2, \ldots, w_n, \ldots) \) of permutations \( w_n \in \mathfrak{S}_n \) coherent with respect to the transition to the induced permutation, \( w'_{n+1} = w_n \). The space of virtual permutations \( \mathfrak{S}_\infty = \lim \mathfrak{S}_n \) is compact and totally disconnected in projective limit topology.

Given a permutation \( g \in \mathfrak{S}_n \), we have \( gw_N \in \mathfrak{S}_N \) for all \( N \geq n \), and the permutations \( gw_n, gw_{n+1}, \ldots \) are coherent. Denoting by \( gw \in \mathfrak{S}_\infty \) the corresponding virtual permutation, we obtain a left action of the group of finite permutations \( \mathfrak{S}_\infty = \lim \mathfrak{S}_n \) in the space \( \mathfrak{S}_\infty \). In a similar way we define the right action \( w \mapsto wg \); it commutes with left shifts \( w \mapsto gw \). We call transformations \( w \mapsto g^{-1}wg \) (where \( g \in \mathfrak{S}_\infty \)) inner automorphisms, or conjugations. We emphasize that this terminology is conditional, since the space \( \mathfrak{S}_\infty \) has no group structure.

The problem of describing measures in the space \( \mathfrak{S}_\infty \) invariant under all conjugations from the group \( \mathfrak{S}_\infty \) is similar to the analogous question on the exchangeable sequences of random variables, i.e., the measures in the space of sequences, invariant under finite permutations. The solution to the latter problem is given by the celebrated de Finetti theorem (see [4], Chap. VII, §4). The first problem was posed (in terms of the so called partition structures) and solved by Kingman [10]. Aldous [5] showed that both problems are essentially equivalent.

As well as in [7], we are interested here in a more special form of the permutational invariance where a measure \( m \) in the space \( \mathfrak{S}_\infty \) is not only invariant under conjugations, but also quasi-invariant upon one-sided actions of the group \( \mathfrak{S}_\infty \).

0.2. Systems of intervals and virtual permutations. In order to define a central measure on the symmetric group \( \mathfrak{S}_n \), one can start with an arbitrary distribution of cycle lengths, and then choose the elements for these cycles from the set \( 1, 2, \ldots, n \) according to the cycle sizes. The element 1 appears in a