LAWS OF LARGE NUMBERS AND THE CENTRAL LIMIT THEOREM FOR SEQUENCES OF COEFFICIENTS OF ROTATIONAL EXPANSIONS

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UDC 519.21

For rotational expansions introduced in [1], conditions under which the law of large numbers, the strong law of large numbers, or the central limit theorem hold for Markov sequences of coefficients, are found. Answers are given in terms of the rate of growth of the quotients $a_n$. Bibliography: 8 titles.

1. In [1], a certain number system was associated with every irrational $\alpha$ from the interval $[0, 1)$; namely, let

$$\alpha = [a_1, a_2, \ldots]$$

be the continued fraction expansion of $\alpha$, let $p_n$, $q_n$ be, respectively, the numerator and denominator of the $n$th convergent of the number $\alpha$ (we set $p_1 = 0$, $p_2 = 1$, $q_1 = 1$, $q_2 = a_1$). We recall that the following relations hold:

$$q_{n+1} = a_n q_n + q_{n-1}, \quad p_{n+1} = a_n q_n + p_{n-1}, \quad n \geq 2.$$

For every positive integer $n$ we put

$$\alpha_n = \lfloor q_n \alpha - p_n \rfloor = (-1)^{n+1} (q_n \alpha - p_n).$$

In [1], it was proved that every $x \in [0, 1)$ can be represented in the form of a convergent series

$$x = \sum_{n=1}^{\infty} x_n \alpha_n \quad (1)$$

with coefficients $x_n$ satisfying the following restrictions on pairs of adjacent coefficients (the so-called Markov restrictions):

$$0 \leq x_n \leq a_n, \quad \text{and if } x_n = a_n, \text{ then } x_{n+1} = 0 \text{ for any } n \in \mathbb{N}.$$

We denote by $\mathcal{X}_\alpha$ the space of all sequences $x_1, x_2, \ldots$ satisfying the restrictions mentioned above and endowed with the topology of coordinate-wise convergence. Formula (1) defines a mapping of the compactum $\mathcal{X}_\alpha$ to the interval $[0, 1)$. Importance of the mapping (1) lies in the fact that it establishes an isomorphism (both topological and metric) between the so-called adic transformation of the compactum $\mathcal{X}_\alpha$ and the transformation $T_\alpha$ of the interval $[0, 1)$ that is defined by the formula $T_\alpha x = x + \alpha \mod 1$, i.e., the rotation of the circle through an irrational angle $\alpha$ (see [1]). This gives grounds to call the expansion (1) of the points of the unit interval a rotational $\alpha$-expansion.

Expansion (1) is unique, except for the points $x$ of the form $\{k\alpha\}$, where $k \in \mathbb{Z}$ (henceforth, $\{t\}$ will stand for the fractional part of a number $t$, and $[t]$ will stand for the integral part of $t$), and the inverse mapping from the interval $[0, 1)$ to the compactum $\mathcal{X}_\alpha$ is defined by the formulas

$$x_1 = \left[ \frac{x}{\alpha} \right], \quad x_n = \left[ \frac{\alpha_{n-1}}{\alpha_n} \left\{ \frac{\alpha_{n-2}}{\alpha_{n-1}} \left\{ \ldots \left\{ \frac{x}{\alpha} \right\} \ldots \right\} \right\} \right], \quad n \geq 2. \quad (2)$$

The image of the Lebesgue measure on the interval $[0, 1)$ under the mapping (2) is the Markov measure $\mu$ on the compactum $\mathcal{X}_\alpha$. The initial distribution and the transition probabilities of the Markov chain $X_\alpha :=$
\((X_\alpha, \mu)\) were computed in [1] as well as the one-dimensional distributions of the random variables \(x_n\). We cite these formulas. The transition probabilities have the form

\[
\mu(x_n = i_n \mid x_{n-1} = i_{n-1}) = \begin{cases} 
\frac{\alpha_n}{\alpha_{n-1}}, & i_{n-1} < a_{n-1}, \quad i_n < a_n \\
\frac{\alpha_{n+1}}{\alpha_{n-1}}, & i_{n-1} < a_{n-1}, \quad i_n = a_n \\
1, & i_{n-1} = a_{n-1}, \quad i_n = 0 \\
0, & \text{otherwise.}
\end{cases}
\] (3)

The one-dimensional distributions are as follows:

\[
\mu(x_n = i_n) = \begin{cases} 
(q_{n-1} + q_n)\alpha_n, & i_n = 0 \\
q_n\alpha_n, & 0 < i_n < a_n \\
q_n\alpha_{n+1}, & i_n = a_n.
\end{cases}
\] (4)

2. The present paper is devoted to the establishment of sufficient conditions met by \(\alpha\) in order that the law of large numbers (the LLN), the strong law of large numbers (the SLLN), or the central limit theorem (the CLT) hold for the Markov chain \(X_\alpha\). These conditions are formulated in terms of quotients of the continued fraction expansion of \(\alpha\). At the end of this paper the conditions obtained above will be compared with similar ones for sequences of independent random variables \((x_n)^\infty\), where \(x_n\) has discrete uniform distribution \(\tau_n\):

\[
\tau_n(x_n = i_n) = 1/a_n, \quad i_n = 0, \ldots, a_n - 1,
\] (5)

and \(a_n \geq 2\). Such a comparison is of interest because for Cantor expansions (naturally generalizing the \(p\)-adic expansions) of the form

\[
x = \sum_{n=1}^\infty \frac{x_n}{a_1 \ldots a_n},
\]

such sequences \((x_n)\) of independent random variables serve as coefficients (see [5]).

Such a comparison shows (see below) that Markov restrictions do not deteriorate too many probabilistic properties of the chain \(X_\alpha\) with respect to the sequence of independent random variables described above.

The author expresses his sincere gratitude to A. M. Vershik for raising the question and helpful discussions.

3. First we prove some technical lemmas.

**Lemma 1.** The mean and variance of the random variables \(x_n\) forming the chain \(X_\alpha\) may be computed by the formulas

\[
E x_n = \kappa_n \left( \frac{a_n - 1}{2} + \frac{\alpha_{n+1}}{\alpha_n} \right), \quad D x_n = \kappa_n (\lambda_n - \varepsilon_n \kappa_n). \quad (6)
\]

Here

\[
\kappa_n = a_n q_n \alpha_n, \quad (7)
\]

\[
\lambda_n = \frac{(a_n - 1)(2a_n - 1)}{6} + \frac{\alpha_{n+1}}{\alpha_n} a_n, \quad \varepsilon_n = \frac{(a_n - 1)^2}{4} + \frac{\alpha_{n+1}(a_n - 1) + \left(\frac{\alpha_{n+1}}{\alpha_n}\right)^2}{\alpha_n}.
\]

**Proof.** Direct inspection.

**Lemma 2.** Let a sequence of reals \(\kappa_n\) be determined by formulas (7) for each \(n\). Then the estimate

\[
1/3 < \kappa_n < 1
\]

holds.

**Proof.** In [4] it is proved that for any \(n \in \mathbb{N}\), the inequalities

\[
\frac{1}{q_n(q_n + q_{n+1})} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}
\]

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