ON THE BEHAVIOR OF SOLUTIONS TO THE NEUMANN PROBLEM IN UNBOUNDED DOMAINS*

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Dedicated to O.A. Oleinik

The paper deals with the uniformly elliptic equation \((a^{ij}(z)u_{x_i})_{x_j} = f(z)\) in an unbounded domain \(\Omega \subset \mathbb{R}^n\) and its solution \(u(x)\) that satisfies the homogeneous Neumann condition. The function \(f\) has a compact support. The domain \(\Omega\) has the following structure: assume that \(\{r_m\}\) is an increasing sequence of positive numbers, \(h_m = r_{m+1} - r_m\), and the ratio \(h_{m+1}/h_m\) lies between positive constants \(C_1\) and \(C_2\). The intersection of \(\Omega\) with the spherical layer between the spheres of radius \(r_m\) and \(r_{m+1}\) with center at the origin satisfies a certain inequality of isoperimetric type. It is shown in this paper that the set of solutions splits up into three classes: (i) the solutions for which \(\lim_{|x| \to \infty} u(x) = \pm \infty\); moreover, it is shown that these limits are attained with nearly the same speed (if \(C_1/C_2 = 1\), then the speed is not less than the exponential one); (ii) the solutions for each of which a constant \(C\) exists such that \(\lim_{|x| \to \infty} u(x) = C\) and \(u(x) - C\) changes its sign for large \(|x|\); here, the convergence to \(C\) is rapid (for \(C_1/C_2 = 1\) this convergence is not slower than the exponential one); (iii) the solutions that do not change their sign for large \(|x|\) and increase or decrease to \(+\infty\) or \(-\infty\), respectively, with low speed (for \(C_1/C_2 = 1\) with a linear speed) (one exception is possible here: a slow convergence to a constant). Bibliography: 7 titles.

INTRODUCTION

1. Assume that the elliptic equation

\[ Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = f(x) \]  

(0.1)

is given in an unbounded domain \(\Omega \subset \mathbb{R}^n\).

The coefficients \(a_{ij}(x)\) and the function \(f(x)\) are subject to the condition

\[ \alpha^{-1} |\xi|^2 \geq \sum_{i,j=1}^{2} a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2, \quad \alpha > 0; \]  

(0.2)

\(f(x) \equiv 0\) outside the ball of a sufficiently great radius \(\tau_0\).

We are interested in the behavior of solutions to (0.1) with the homogeneous Neumann condition on the boundary:

\[ \frac{\partial u}{\partial v} \bigg|_{\partial \Omega} = 0. \]  

(0.3)

The purpose of this work is to study the connection between the behavior of the solution to problem (0.1), (0.3) at infinity and the geometry of the domain \(\Omega\).

By the solution to problem (0.1), (0.3), we mean the function

\[ u \in C^2_\text{loc}(\Omega) \cap C^1_\text{loc}(\bar{\Omega}) \]  

(0.4)

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that satisfies Eq. (0.1) and condition (0.3) in the classical sense.

We assume that \( a_{ij}(x), f(x), \) and the boundary of the domain have a certain smoothness.

**Remark.** The theorems proved in this paper do not depend on the degree of smoothness of the coefficients \( a_{ii} \), of the function \( f(x) \) and of \( \partial \Omega \), and, therefore, they are valid for generalized solutions, which are defined by an integral identity as well. For simplicity, we restrict ourselves in this paper to the case of the classical solution.

The question of the behavior of solutions to (0.1) in an unbounded domain has been treated in a comparatively small number of works, of which we mention [1–5].

This work, in a sense, a continuation of [1, 3], where it was shown that if an unbounded domain is close to a half-cylinder, then the solution to problem (0.1), (0.3) behaves at infinity in one of the following three ways: it either changes its sign and grows rapidly, or changes its sign and decreases rapidly, or has intermediate asymptotics (linear in the case of a cylinder). This partition of solutions into three classes is called a trichotomy.

In this work, we show that a similar partition into three classes takes place for a considerably wider class of domains than a half-cylinder. The conditions on the domain, under which the trichotomy of solutions to the Neumann problem is valid, are connected with the isoperimetric properties of the domain.

1. **THE ISOPERIMETRIC INEQUALITY**

**Definition.** Assume that \( G \subset \mathbb{R}^n \) is a bounded domain.

We say that the inner isoperimetric inequality with a constant \( a > 0 \) is valid in \( G \) if for any subdomain \( G' \subset G \) the inequality

\[
\mu_{n-1}(\partial G' \cap G) \geq \min \left( \frac{\mu_n G'}{(n-1)/n}, \frac{\mu_n (G \setminus G')}{(n-1)/n} \right)
\]

holds, where \( \mu_k E \) is the \( k \)-dimensional Hausdorff measure of the set \( E \subset \mathbb{R}^n \) (so that \( \mu_n E \) is the \( n \)-dimensional Lebesgue measure of the set \( E \subset \mathbb{R}^n \)).

**Lemma 1.1** (on the inner isoperimetric inequality). Assume that the inner isoperimetric inequality with the constant \( a > 0 \) is fulfilled for a bounded domain. Then there exists a constant \( b > 0 \) that depends on \( a \) and on the dimension \( n \) of the space such that

\[
diam(G) \leq b(\mu_n G)^{1/n}.
\]  

**Proof.** The statement is obvious for \( n = 1 \). Therefore, we assume that \( n \geq 2 \). Further, it is sufficient to consider the case where

\[
\mu_n G = 1.
\]  

(1.3)

To show this, we make the change of variables

\[
\tilde{x} = \frac{x}{(\mu_n G)^{1/n}}.
\]

Assume that \( G \) and \( G' \) turn into \( \tilde{G} \) and \( \tilde{G}' \), respectively, so that

\[
\mu_n \tilde{G} = 1.
\]

For \( \tilde{G} \) and \( \tilde{G}' \), (1.1) is fulfilled with the same constant. Therefore, we will assume that (1.3) is valid.

Assume that

\[
diam \ G = 2\beta
\]  

(1.4)

and

\[
\beta > 1.
\]  

(1.5)

Otherwise, the statement of the lemma is valid.

There exist two points \( z_1 \) and \( z_2 \) on \( \partial G \) such that \( |z_1 - z_2| = 2\beta \).

Assume that \( \pi_1 \) and \( \pi_2 \) are the hyperplanes of support to \( G \) at the points \( z_1 \) and \( z_2 \) respectively.

If \( \pi_1 \) and \( \pi_2 \) are defined in a unique way, then they are parallel and the segment \([x_1, x_2]\) is orthogonal to them. If one of these hyperplanes or both of them are not unique, then it is possible to choose them so that they would be orthogonal to the segment \([x_1, x_2]\).

Assume that \( x_0 \) is a point of the segment \([x_1, x_2]\) such that the hyperplane \( \pi_0 \) that goes through \( x_0 \) and is orthogonal to this segment divides \( G \) into two parts of equal volumes.