Asymptotic Analysis of Certain Classes of Singularly Perturbed Problems on the Semiaxis

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ABSTRACT. We propose a new method for asymptotic integration of certain classes of singularly perturbed Cauchy problems on the semiaxis for nonhomogeneous systems of linear ordinary differential equations; this method is a further development of the ideas of the regularization method. This method enables us to prove the existence of a unique bounded (as $\varepsilon \to +0$) solution of such problems and leads to a simpler and more constructive algorithm for obtaining the asymptotic expansion of the solution and singling out all of its singularities in closed analytic form (including the critical case in which the spectral points of the limit operator may touch the imaginary axis). The proposed method supplements and sharpens earlier results.

KEY WORDS: asymptotic integration of singularly perturbed Cauchy problems, nonhomogeneous systems of linear ordinary differential equations, regularization method.

A new method is proposed for asymptotic integration of certain classes of singularly perturbed Cauchy problems on the semiaxis $\mathbb{R}^+$ for nonhomogeneous systems of linear ordinary differential equations; this method is a further development of the ideas of the regularization method [1].

In contrast to the well-known method [2-5], our method, in conjunction with the results from [6-9], enables us to prove the existence of a unique bounded (as $\varepsilon \to +0$) solution of such problems and leads to a simpler and more constructive algorithm for obtaining the asymptotic expansion of the solution and singling out all of its singularities in closed analytic form.

Consider the following singularly perturbed Cauchy problem on the semiaxis $\mathbb{R}^+$:

$$\varepsilon \frac{dy}{dx} = A(x, \varepsilon)y + f(x), \quad y(0, \varepsilon) = \alpha,$$  \hspace{1cm} (1)

where the matrix series $A(x, \varepsilon) = \sum_{k=0}^{\infty} A_k(x)\varepsilon^k$ is absolutely and uniformly convergent on the semiaxis $\mathbb{R}^+$ for $|\varepsilon| < \varepsilon_0$, the functions $f(x)$ and $A_k(x) \in C^{N+1}(\mathbb{R}^+)$ are bounded on the semiaxis $\mathbb{R}^+$ together with the appropriate number of derivatives, i.e.,

$$||f^{(p)}(x)|| \leq C_p, \quad ||A_k^{(p)}(x)|| \leq C_{kp}, \quad (p = 0, \ldots, N+1, \quad k \geq 0, \quad x \in \mathbb{R}^+),$$  \hspace{1cm} (2)

and the matrix $A_0(x)$ has a simple nonzero stable spectrum $\{\lambda_{0j}(x)\}_{j=1}^n$, i.e.,

$$\sigma_{jk}(x) \equiv \lambda_{0j}(x) - \lambda_{0k}(x) \neq 0, \quad |\lambda_{0j}(x)| \geq \delta_0 > 0.$$  \hspace{1cm} (3)

Unlike singularly perturbed Cauchy problems of the form (1) on a finite interval, the singularities of the solution of a similar problem on the semiaxis are determined in the general case not only by the spectrum of the matrix $A_0(x)$, but also by the structure of some integrals that are unbounded as $x \to +\infty$.

Before proceeding with these assertions, let us first use the shift transformation

$$y(x, \varepsilon) = q(x, \varepsilon) + w(N)(x, \varepsilon) \quad \left( w(N)(x, \varepsilon) = \sum_{k=0}^{N} w_k(x)\varepsilon^k \right)$$

to obtain the following almost homogeneous Cauchy problem for the function $q(x, \varepsilon)$:

$$\varepsilon \frac{dq}{dx} = A(x, \varepsilon)q + \varepsilon^{N+1}b_{N1}(x, \varepsilon), \quad q(0, \varepsilon) = \alpha - w(N)(0, \varepsilon) \equiv \beta(\varepsilon),$$  \hspace{1cm} (4)
where the vector function $w_{(N)}(x, \varepsilon)$ regular with respect to $\varepsilon$ can be uniquely determined (by substitution into Eq. (1)) from the relations

$$0 = A_0(x)w_0(x) + f(x), \quad \frac{d w_{k-1}}{d x} = A_0(x)w_k(x) + \cdots + A_k(x)w_0(x) \quad (k = 1, \ldots, N);$$

here

$$\|w_k^{(p)}(x)\| \leq C_{1kp}, \quad (k = 0, \ldots, N, \quad p = 0, \ldots, N + 1, \quad x \in \mathbb{R}^+). \quad (5)$$

For an arbitrary matrix $A = \{a_{jk}\}_1^n$, let

$$\overline{A} = \text{diag}\{a_{11}, \ldots, a_{nn}\}, \quad \overline{\overline{A}} = A - \overline{A};$$

this notation will be used in the following.

**Theorem 1.** For problem (1), let conditions (2) and (3) and inequality (5) be satisfied, and let the spectrum $\{\lambda_{0j}(x)\}_1^n$ of the matrix $A_0(x)$ satisfy the inequalities

$$\text{Re} \lambda_{0j}(x) \leq -\delta_0 < 0 \quad (j = 1, \ldots, n, \quad x \in \mathbb{R}^+);$$

moreover, let the matrices $S_0(x), \Lambda_0(x),$ and $B_k(x) \ (k \geq 1)$, where

$$S_0^{-1}(x)A_0(x)S_0(x) = \Lambda_0(x) = \text{diag}\{\lambda_{01}(x), \ldots, \lambda_{0n}(x)\},$$

$$B(x, \varepsilon) = S_0^{-1}(x)\left(A(x, \varepsilon)S_0(x) - \varepsilon \frac{dS_0}{dx}\right) = \Lambda_0(x) + \sum_{k=1}^{\infty} B_k(x)\varepsilon^k,$$

satisfy the following estimates:

$$\|S_0(x)\| \leq C_0, \quad \|B_q^{(p)}(x)\| \leq C_{qP}, \quad \frac{\left(B_q^{(p)}(x)\right)}{\sigma_{jk}(x)} \leq C_{qP} \quad (q \geq 1, \quad p = 0, \ldots, N + 1, \quad j \neq k, \quad j, k = 1, \ldots, n, \quad x \in \mathbb{R}^+). \quad (6)$$

Then for sufficiently small $\varepsilon > 0$ and any $x \in \mathbb{R}^+$ there exists a unique bounded (as $\varepsilon \to 0^+$) solution $y(x, \varepsilon)$ of problem (1) of the following asymptotic form:

$$y(x, \varepsilon) \sim Y_{(N)}(x, \varepsilon) = S_0(x)\left(E + \sum_{k=1}^{N} \overline{H}_k(x)\varepsilon^k\right)\exp\left(\frac{1}{\varepsilon} \int_0^x \Lambda_{(N)}(t, \varepsilon) \, dt\right) + w_{(N)}(x, \varepsilon) \quad (7)$$

$$\left(\Lambda_{(N)}(x, \varepsilon) = \sum_{k=0}^{N} \Lambda_k(x)\varepsilon^k\right)$$

(the matrix functions $\overline{H}_k(x)$ and $\Lambda_k(x) \ (k = 1, \ldots, N$) are uniquely determined by the algorithm); furthermore, the following estimate is valid on the entire semiaxis $\mathbb{R}^+$:

$$\|y(x, \varepsilon) - Y_{(N)}(x, \varepsilon)\| \leq K\varepsilon^{N+1}, \quad (8)$$

where the constant $K$ is independent of $\varepsilon$. 

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