SOLUTION OF TWO-DIMENSIONAL PROBLEMS OF ELASTICITY AND THERMOELASTICITY FOR A RECTANGULAR REGION

V. M. Vigak

We propose a separation-of-variables method for the biharmonic equation and construct a complete system of orthogonal functions for constructing exact solutions in the form of non-periodic trigonometric series for two-dimensional problems of elasticity and thermoelasticity for a rectangular region.

Methods of solving problems of elasticity for bounded bodies with corners have been discussed in [4–6]. The absence of exact closed-form solutions for elasticity problems for such bodies is connected with the separation of variables in the key biharmonic differential equations. In the present paper we propose a method of constructing exact solutions of two-dimensional problems of elasticity and thermoelasticity in the stresses for a rectangular region under prescribed forces on the boundary. The method is based on the integration of the differential equations of equilibrium, making it possible to express two of the components of the stress tensor in terms of one of the normal components and equivalently replacing the eight boundary conditions for the different components by six conditions for a single component of the normal stresses. This makes it possible to separate the variables in the key biharmonic equation and to give the solution of the problems of elasticity and thermoelasticity just posed in the form of expansions in nonperiodic trigonometric series using a complete system of orthogonal functions of the corresponding nonclassical spectral problem.

We consider the two-dimensional quasi-static problem of thermoelasticity for a rectangular region \( G = \{(x, y) \in [-1, 1] \times [-a, a]\} \) with prescribed forces on the boundary. From the determinate system of equations that describes the thermoelastic state in a homogeneous isotropic body we take the equations of equilibrium

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \quad (x, y) \in G, \tag{1}
\]

and the equation of continuity in the stresses [1, 2]

\[
\Delta^2 \sigma_v = -\frac{\partial^2}{\partial x^2} \Delta T, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad (x, y) \in G, \tag{2}
\]

with boundary conditions

\[
\sigma_x \bigg|_{x=1} = p_1(y), \quad \sigma_x \bigg|_{x=-1} = p_2(y), \quad \sigma_{xy} \bigg|_{x=1} = p_3(y), \quad \sigma_{xy} \bigg|_{x=-1} = p_4(y), \tag{3}
\]

\[
\sigma_y \bigg|_{y=a} = q_1(x), \quad \sigma_y \bigg|_{y=-a} = q_2(x), \quad \sigma_{xy} \bigg|_{y=a} = q_3(x), \quad \sigma_{xy} \bigg|_{y=-a} = q_4(x). \tag{4}
\]

Here and below the components of the stress \( \sigma_{ij} \) have dimension of temperature \( \sigma_{ij} = \frac{1-\nu}{\alpha E} \sigma^{*}_{ij} \), where \( E \) is the elastic modulus, \( \nu \) and \( \alpha \) are the Poisson coefficient and the coefficient of linear expansion, and \( \sigma^{*}_{ij} \) is the stress of natural dimension, \( i, j = x, y \).

The equations of equilibrium (1) are required to hold also on the boundary of the region \( G \), that is, when we take account of the boundary conditions (3) and (4), the normal components of the stresses must also satisfy the following boundary conditions

\[
\frac{\partial \sigma_x}{\partial x} \bigg|_{x=1} = -\frac{dp_3}{dy}, \quad \frac{\partial \sigma_x}{\partial x} \bigg|_{x=-1} = -\frac{dp_4}{dy}, \tag{5}
\]

\[
\frac{\partial \sigma_y}{\partial y} \bigg|_{y=a} = -\frac{dq_3}{dx}, \quad \frac{\partial \sigma_y}{\partial y} \bigg|_{y=-a} = -\frac{dq_4}{dx}. \tag{6}
\]
We solve the second of Eqs. (1) for the components \( \sigma_{xy} \), applying the boundary conditions (3). As a result we obtain

\[
2\sigma_{xy} = p_3 + p_4 - \int_{-1}^{1} \frac{\partial \sigma_{xy}}{\partial y} \text{sgn}(x - \eta) \, d\eta, \quad \text{sgn} x = \begin{cases} 
1, & x > 0, \\
0, & x = 0, \\
-1, & x < 0.
\end{cases}
\]  
(7)

This equation is also valid at \( x = \pm 1 \), from which it follows that

\[
\frac{d}{dy} \int_{-1}^{1} \sigma_{xy} \, dx = p_4 - p_3.
\]

By integrating this equation, applying the boundary conditions (4) for the stresses \( \sigma_y \), we obtain

\[
2 \int_{-1}^{1} \sigma_y \, dx = \int_{-1}^{1} (q_1 + q_2) \, dx + \int_{-a}^{a} (p_4 - p_3) \text{sgn}(y - \xi) \, d\xi,
\]  
(8)

so that for \( y = \pm a \) we obtain a necessary condition for equilibrium

\[
\int_{-1}^{1} (q_1 - q_2) \, dx + \int_{-a}^{a} (p_3 - p_4) \, dy = 0.
\]  
(9)

Solving similarly the first of Eqs. (1) for the stresses \( \sigma_{xy} \), we find the relations

\[
2\sigma_{xy} = q_3 + q_4 - \int_{-a}^{a} \frac{\partial \sigma_{xy}}{\partial x} \text{sgn}(y - \xi) \, d\xi,
\]  
(10)

\[
2 \int_{-a}^{a} \sigma_x \, dy = \int_{-a}^{a} (p_1 + p_2) \, dy + \int_{-1}^{1} (q_4 - q_3) \text{sgn}(x - \eta) \, d\eta
\]  
(11)

and the equilibrium condition

\[
\int_{-a}^{a} (p_1 - p_2) \, dy + \int_{-1}^{1} (q_3 - q_4) \, dx = 0.
\]  
(12)

Integrating the first of Eqs. (1) with respect to \( x \) and the second with respect to \( y \), we obtain

\[
2\sigma_x = p_1 + p_2 - \int_{-1}^{1} \frac{\partial \sigma_{xy}}{\partial x} \text{sgn}(x - \eta) \, d\eta,
\]  
(13)

\[
2 \int_{-1}^{1} \sigma_{xy} \, dx = \int_{-1}^{1} (q_3 + q_4) \, dx + \int_{-a}^{a} (p_2 - p_1) \text{sgn}(y - \xi) \, d\xi.
\]  
(14)

\[
2\sigma_y = q_1 + q_2 - \int_{-a}^{a} \frac{\partial \sigma_{xy}}{\partial x} \text{sgn}(y - \xi) \, d\xi,
\]  
(15)

\[
2 \int_{-a}^{a} \sigma_{xy} \, dy = \int_{-a}^{a} (p_3 + p_4) \, dx + \int_{-1}^{1} (q_2 - q_1) \text{sgn}(x - \eta) \, d\eta.
\]  
(16)

Integrating the expression (14) with respect to \( y \) and (16) with respect to \( x \) between the corresponding limits, and equating the right-hand sides, we obtain the equality

\[
\int_{-1}^{1} x(q_1 - q_2) \, dx + \int_{-a}^{a} (p_3 + p_4) \, dy = \int_{-a}^{a} y(p_1 - p_2) \, dy + a \int_{-1}^{1} (q_3 + q_4) \, dx.
\]  
(17)