OPTIMIZATION OF THE TWO-DIMENSIONAL TEMPERATURE FIELDS IN INHOMOGENEOUS BODIES WITH CONSTRAINTS ON THE PHASE COORDINATES

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We state a problem of speed-optimal control of the two-dimensional nonstationary temperature regime in inhomogeneous bodies with constraints on the control (the temperature of the heating medium) and the phase coordinates (the temperature of the body, the temperature range, and the heat flux on the surface of the body), and we propose a method for solving the problem numerically. We give as an example the computation of the optimal control of heating of a hollow infinite cylinder under constraints on the temperature range.

Let the temperature $T(x, y, t)$ in a body during heating satisfy the following heat equation:

$$
\frac{\partial}{\partial x} \left[ \lambda(x, y) \frac{\partial T}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \lambda(x, y) \frac{\partial T}{\partial y} \right] - H(x, y)T = c(x, y)\rho(x, y) \frac{\partial T}{\partial t},
$$

$$
(x, y, t) \in D = V \times [0, t_1],
$$

and the boundary conditions

$$
T(x, y, 0) = f(x, y) \quad (x, y) \in \overline{V},
$$

$$
\lambda(x, y) \frac{\partial T}{\partial n} - \alpha(x, y)(T - q) = 0, \quad (x, y, t) \in \Gamma = \Gamma \times [0, t_1].
$$

Here $V$ is the domain of definition of the variables $x$ and $y$, bounded by the surface $\Gamma$; $\overline{V} = V \cup \Gamma$; $n$ is the unit normal to the surface $\Gamma$; $t$ is time; $\lambda(x, y) \in C(\overline{V})$ is the coefficient of thermal conductivity of the material; $c(x, y)$ is the specific mass heat capacity; $\rho(x, y)$ is the density of the material; $f(x, y) \in C(\overline{V})$ is the initial temperature distribution in the body; $\alpha(x, y)$ is the coefficient of heat transfer.

If $H(x, y) \equiv 0$ in Eq. (1), the boundary-value problem (1)--(3) describes the heating of a body whose temperature field depends only on the two spatial coordinates. When $H(x, y) \neq 0$, the problem (1)--(3) describes the heating of a thin-walled shell [5], where $T(x, y, t)$ is the integral-mean average temperature over the thickness of the shell and $H(x, y)$ is the coefficient of heat transfer of its surfaces with the surrounding medium at temperature zero.

Suppose the body is heated by varying the temperature of the surrounding medium $q(x, y, t)$, which we choose as the control function. In the heating process it is necessary to take account of the constraints on the parameters of the thermal process, which can be given in the general form by the inequality

$$
FT \leq \delta(x, y, t), \quad (x, y, t) \in D.
$$

Here $F$ is the operator that determines the constraining parameters and $\delta \in C(D)$ are their assigned limiting admissible values. Such parameters are usually taken [2] to be:

a) the maximal temperature of the body

$$
FT = \max_{(x, y) \in \overline{V}} T(x, y, t);
$$

b) the temperature range

$$
FT = \max_{(x, y) \in \overline{V}} T(x, y, t) - \min_{(x, y) \in \overline{V}} T(x, y, t);
$$
c) the maximal gradient of the temperature field on the surface of the body

\[ FT = \max_{(x,y) \in \Gamma} \frac{\partial T(s,y,t)}{\partial n}. \]

We consider the following optimization problem. Determine a control \( q = \omega \) bounded above by a given function \( l(x,y,t) \):

\[ q(x,y,t) \leq l(x,y,t), \quad (x,y,t) \in S, \tag{5} \]

which takes the body from the initial state (2) to a final state with a prescribed temperature \( T_0^* \) at some point \((x_*,y_*) \in \mathcal{V}: \)

\[ T_0^* = T(x_*,y_*,t_0) \tag{6} \]

in minimal time \( t_0 \in [0,t_1] \) subject to the constraint (4).

According to [3] the control \( \omega(x,y,t) \) is speed-optimal if at each point of the surface \( \Gamma \) for \( t \in [0,t_0] \) it either equals its limiting possible value

\[ \omega(x,y,t) = l(x,y,t), \quad (x,y,t) \in S_t = \Gamma \times [0,t_0], \tag{7} \]

or causes the constraining parameters to assume their limiting admissible values

\[ FT_\omega = \delta(x,y,t), \quad (x,y,t) \in S_\delta = \Gamma_\delta \times [0,t_0], \tag{8} \]

where \( T_\omega \) is a solution of the boundary-value problem (1)–(3) for \( q = \omega \); \( \Gamma_\delta \subseteq \Gamma \). Then to determine the speed-optimal temperature regime \( T_\omega \) it suffices to find a solution of Eq. (1) that satisfies the boundary conditions (2) and (3) for \( q = \omega \) on the surface \( S_t \) and also condition (8) on the surface \( S_\delta \), where we find the control on \( S_\delta \) by substituting the solution \( T_\omega \) already found into relation (3).

Since the surfaces \( S_t \) and \( S_\delta \) are not known in advance and depend on the parameters and functions being sought, as well as the temperature field, this optimization problem is in general nonlinear.

To solve the problem we use the finite-difference method in the direction variables [4]. To this end we introduce a grid into the region \( \mathcal{D} \) with nodes \( \{(x_i,y_j,t_k), 0 \leq i \leq I, 0 \leq j \leq J, 0 \leq k \leq K\} \), where depending on the geometry of the region \( \mathcal{V} \) the grid may be chosen to be nonuniform on \( x \) and \( y \). Following the procedure proposed in [4], we write the finite-difference analog of the boundary-value problem (1)–(3), (7), (8) as

\[
\frac{2}{\tau} \left( T_{ij}^{(k+1/2)} - T_{ij}^{(k)} \right) = L_1 T_{ij}^{(k+1/2)} + L_2 T_{ij}^{(k)}, \quad (x_i,y_j,t_k) \in D; \tag{9}
\]

\[
T_{ij}^{(0)} = f_{ij}, \quad (x_i,y_j) \in \mathcal{V}; \tag{10}
\]

\[
F_{ij} T = \frac{\alpha_{ij}}{\lambda_{ij}} \left[ T_{ij}^{(k)} - \omega_{ij}^{(k)} \right], \quad (x_i,y_j) \in A^{(k)} \subset Z; \tag{11}
\]

\[
F_{ij} T = \delta_{ij}^{(k)}, \quad (x_i,y_j) \in B^{(k)} \subset Z. \tag{12}
\]

Here \( T_{ij}^{(k+1/2)} \) and \( T_{ij}^{(k)} \) are respectively the temperature \( T(x,y,t) \) at the point with coordinates \( (x_i,y_j) \in \mathcal{V} \) in the half-integer time layer \( t_{k+1/2} = t_k + \tau /2 \) and the integer time layer \( t_k = k \cdot \tau \); \( \tau \) is the step with respect to time; \( f_{ij}, \alpha_{ij}, \lambda_{ij}, \omega_{ij}^{(k)} \) and \( \delta_{ij}^{(k)} \) are the values of the functions \( f, \alpha, \lambda, \omega, \) and \( \delta \) at the nodes of the grid \( \{(x_i,y_j,t_k)\} \); \( Z \) is the set of points of the finite-difference approximation of the boundary \( \Gamma \) of the region \( \mathcal{V} \); \( A^{(k)} \) and \( B^{(k)} \) are the sets of nodes of the approximation of the boundaries \( \Gamma_t \) and \( \Gamma_\delta \) at time \( t = t_k \), so that \( A^{(k)} \cup B^{(k)} = Z \); \( F_{ij} \) and \( \Phi_{ij} \) are the finite-difference analogs of the operators \( F \) and \( \partial/\partial n \). The form of the operators \( L_1 \) and \( L_2 \) is given in [1, 4].

In order to find the numerical solution of the problem (1)–(3), (7), (8) at the time \( t = t_{k+1} \), it is necessary first to use the known values of the temperature \( T_{ij}^{(k)} \), \( i = 0,I, j = 0,J \), at the \( k \)th time layer to