ON IRREDUCIBLE FACTORIZATIONS OF RATIONAL MATRICES AND THEIR APPLICATIONS

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This paper is an extension of our studies of the computational aspects of spectral problems for rational matrices pursued in previous papers. Methods of solution of spectral problems for both one-parameter and two-parameter matrices are considered. Ways of constructing irreducible factorizations (including minimal factorizations with respect to the degree and size of multipliers) are suggested. These methods allow us to reduce the spectral problems for rational matrices to the same problems for polynomial matrices. A relation is established between the irreducible factorization of a one-parameter rational matrix and its irreducible realization used in system theory. These results are extended to the case of two-parameter rational matrices. Bibliography: 15 titles.

INTRODUCTION

Let \( R(\lambda) = \{r_{ij}(\lambda)\} \) and \( R(\lambda, \mu) = \{r_{ij}(\lambda, \mu)\} \) be one- and two-parameter rational \( m \times n \) matrices, i.e., matrices whose entries are rational functions

\[
\begin{align*}
  r_{ij}(\lambda) &= \frac{f_{ij}(\lambda)}{g_{ij}(\lambda)}, \\
  r_{ij}(\lambda, \mu) &= \frac{f_{ij}(\lambda, \mu)}{g_{ij}(\lambda, \mu)},
\end{align*}
\]

where \( f_{ij}(\lambda), g_{ij}(\lambda) \) and \( f_{ij}(\lambda, \mu), g_{ij}(\lambda, \mu) \) are scalar polynomials in one and two variables, respectively. We denote by \( \rho \) the rank of the matrix \( R : \rho \leq \min(m, n) \). If \( m = n \) and \( \det R \equiv 0 \), then the matrix \( R \) is called regular; otherwise, \( R \) is called a singular matrix.

In this paper, various forms of factorizations of the matrices \( R(\lambda) \) and \( R(\lambda, \mu) \) and their applications to the solution of spectral problems are considered. The paper consists of six sections. A classification of different factorizations of rational matrices \( R(\lambda) \) and \( R(\lambda, \mu) \) of a general form is suggested in Sec. 1. Auxiliary (base) factorizations and methods of their construction are considered. A classification of the so-called irreducible factorizations is proposed. Invariant spectral properties are proved for the factors of irreducible factorizations constructed in different ways for the same rational matrix. Algorithms for constructing two-factor irreducible factorizations for \( R(\lambda) \) and \( R(\lambda, \mu) \) are presented in Sec. 2. We observe that an algorithm for constructing the irreducible factorizations of \( R(\lambda) \) was described in [4], where this factorization was called minimal. In view of the classification suggested in this paper, we classify the factorization from [4] as an irreducible, but not minimal, factorization. In Secs. 3 and 4, for \( R(\lambda) \) and \( R(\lambda, \mu) \) the so-called minimal irreducible factorizations are considered. As irreducible ones, these factorizations have factors of minimal size and degree.

Spectral problems for rational matrices are formulated in Sec. 5. In that section, we prove that these problems may be reduced to spectral problems for polynomial matrices by using irreducible and minimal factorizations.

In Sec. 6, a relation between the irreducible factorizations under consideration and the so-called irreducible realizations in system theory [7, 8] is established for a one-parameter rational matrix \( R(\lambda) \). The results obtained are extended to the case of two-parameter rational matrices.

§ 1. CLASSIFICATION OF SOME FACTORIZATIONS OF RATIONAL MATRICES AND THEIR PROPERTIES

Let \( R(\lambda) \) and \( R(\lambda, \mu) \) be rational \( m \times n \) matrices of rank \( \rho \). Consider factorizations of the form

\[
\begin{align*}
  R(\lambda) &= K^{-1}(\lambda)D(\lambda), \\
  R(\lambda, \mu) &= K^{-1}(\lambda, \mu)D(\lambda, \mu),
\end{align*}
\]

*Sometimes, when writing one- and two-parameter matrices, we will omit the parameters if the reasoning is valid in either case.

\[ R(\lambda) = D(\lambda)K^{-1}(\lambda), \quad (1.3) \]
\[ R(\lambda, \mu) = D(\lambda, \mu)K^{-1}(\lambda, \mu), \quad (1.4) \]
satisfying the following properties:
(a) \( K \) and \( D \) are polynomial matrices of relevant sizes;
(b) \( K \) is a regular matrix.

We shall refer to factorizations (1.1)--(1.4) as base ones. In what follows, these factorizations will play an auxiliary role and provide a basis for the construction of other factorizations of rational matrices.

If the original rational matrix \( R \) is not given in one of the forms (1.1) or (1.3) and (1.2) or (1.4), then, as a base factorization for \( R \), we can take, for example, the factorization with a regular matrix \( K \) of the form

\[ K = kI_m \quad \text{or} \quad K = \text{diag} \{ k_1, \ldots, k_m \} \]

for (1.1) and (1.2), and

\[ K = kI_n \quad \text{or} \quad K = \text{diag} \{ k_1, \ldots, k_n \} \]

for (1.3) and (1.4). Here, \( k \) is a scalar polynomial, namely, the least common multiple (LCM) of the denominators of all the entries of \( R \); \( k_i, i = 1, \ldots, m, (\overline{k_i}, i = 1, \ldots, n) \) is the LCM of the denominators of the entries of the \( i \)th row (the \( i \)th column), respectively, of the matrix \( R \). For methods of computing the LCM of scalar polynomials in one and two variables, see [15].

The following factorizations of a rational \( m \times n \) matrix \( R \) of rank \( \rho \) are called irreducible.

1. The factorization
\[ R = ST^{-1} \quad (1.5) \]
satisfying the following properties:
(a) \( S, T \) are polynomial \( m \times n \) and \( n \times n \) matrices, respectively;
(b) \( T \) is a regular matrix;
(c) \( M := [S, T]^B \) is a polynomial \( (m + n) \times n \) matrix* which does not possess
   • a finite spectrum in the case \( R := R(\lambda) \);
   • a finite continuous spectrum in the case \( R := R(\lambda, \mu) \).

2. The factorization
\[ R = T^{-1}S \quad (1.6) \]
satisfying the following properties:
(a) \( T \) and \( S \) are \( m \times m \) and \( m \times n \) polynomial matrices, respectively;
(b) \( T \) is a regular matrix;
(c) \( M := [S, T] \) is a polynomial \( m \times (n + m) \) matrix which does not possess
   • a finite spectrum in the case \( R := R(\lambda) \);
   • a finite continuous spectrum in the case \( R := R(\lambda, \mu) \).

3. The factorization
\[ R = ST^{-1}V \quad (1.7) \]
satisfying the following properties:
(a) \( S, T, \) and \( V \) are polynomial \( m \times q, q \times q, \) and \( q \times n \) \((q \geq \rho)\) matrices, respectively;
(b) \( T \) is a regular matrix;
(c) the polynomial \( (m + q) \times q \) and \( q \times (q + n) \) matrices \([S, T]^B\) and \([T, V]\), respectively, do not possess
   • a finite spectrum in the case \( R := R(\lambda) \);
   • a finite continuous spectrum in the case \( R := R(\lambda, \mu) \).

Remark. The base factorizations (1.1)--(1.4) considered above are not, as a rule, irreducible.

*Henceforth, the symbol \([A, C]^B\) denotes block transposition, i.e., \([A, C]^B \equiv \begin{bmatrix} A \\ C \end{bmatrix} \).