NONHOMOGENEOUS WARING EQUATIONS

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It is proved that for an arbitrary positive integer \( k \) the equation \( n = x^2 + y^2 + z^3 + u^3 + v^4 + w^{14} + t^{4k+1} \) has a positive integer solution for all sufficiently large \( n \). Bibliography: 6 titles.

The nonhomogeneous Waring equations have the form

\[ n = x_1^{k_1} + \ldots + x_r^{k_r}, \]

where \( 2 \leq k_1 \leq \ldots \leq k_r \). It would be natural to expect that the number of positive solutions of this equation is of order \( n^{1/k_1+\ldots+1/k_r-1} \) for large \( n \). Therefore, it is necessary to require that \( 1/k_1 + \ldots + 1/k_r > 1 \). Vaughan [1] conjectured that Eq. (1) is solvable for all sufficiently large \( n \) if, in addition to the last inequality, assumptions like the solvability of the corresponding congruence with respect to all modules (which, obviously, are necessary) are fulfilled. Many mathematicians studied this equation for various combinations of exponents \( k_i \). With the exception of the Gauss theorem on the three squares, all the obtained results apply to the case where \( 1/k_1 + \ldots + 1/k_r \geq 2 \). This sum takes the smallest value in the theorem by Yu. V. Linnik [2] on the solvability of the equation

\[ n = x^2 + y^2 + z^3 + u^3 + v^3 \]

for all sufficiently large \( n \) and in [3], where the equations

\[ n = x^2 + y^2 + z^3 + u^3 + v^a + w^b \]

are studied for arbitrary positive \( a \) and \( b \) satisfying the condition \( 1/a + 1/b > 1/3 \).

In the present paper we prove that for arbitrary positive integer \( k \) the equation

\[ n = x^2 + y^2 + z^3 + u^3 + v^4 + w^{14} + t^{4k+1} \]

is solvable for all sufficiently large \( n \). Here it is clear that the sum of the inverse values of exponents is less than 2 if \( k \) is sufficiently large.

When studying Eq. (2), Linnik reduced the problem of its solvability to the problem of representability of large numbers by ternary quadratic forms. Note that his theorem on seven cubes [4] is obtained along the same lines.

A similar approach is used in the present paper. The main difference in our case is that we have to do with a ternary quadratic form whose discriminant increases together with the represented number. For this reason, the used estimates of the Fourier coefficients of cusp forms of weight 3/2 should be uniform with respect to the level.

The modifications which we made in Linnik's approach allow us to consider not only Eq. (3), but also other equations of the same type.

The limit case for this approach is the equation

\[ n = x^2 + y^2 + z^3 + u^3. \]

The corresponding quadratic form has discriminant of order \( \sqrt[4]{n} \). On the other hand, there is a conjecture stating that the Fourier coefficient of the cusp form of weight 3/2 occurring in the corresponding theta-series admits the estimate \( \ll n^{1/4} \), which is uniform in level. Though this conjecture is far beyond the known results, it is nevertheless quite plausible.

Now let us turn to the statements of our main results.

Theorem. The equation
\[ n = x^2 + y^2 + z^3 + u^3 + v^4 + w^{14} \] (4)
is solvable in positive integers if \( n \equiv 1 \pmod{5} \) and \( n \) is sufficiently large.

Corollary. Equation (3) is solvable in nonnegative integers for all sufficiently large \( n \).

The latter statement immediately follows from the theorem because \( t^{4k+1} \equiv t \pmod{5} \) and hence for some \( t \) such that \( 0 \leq t \leq 4 \) we have \( n - t^{4k+1} \equiv 1 \pmod{5} \).

To prove the theorem, we need a lemma on the representability of numbers by positive-definite ternary quadratic forms with large discriminant.

Lemma. Let \( p \) be a prime such that \( (m, 6p) = 1 \) and let the congruence
\[ m \equiv x^2 + y^2 + 6pz^2 \pmod{16} \]
have a solution. Let \( r(m) \) denote the number of solutions of the equation
\[ m = x^2 + y^2 + 6pz^2. \] (5)

Let \( m > p^{21.5} \). Then for sufficiently large \( p \) we have \( r(m) > 0 \).

Proof. Let us represent the theta-series of the quadratic form \( x^2 + y^2 + 6pz^2 \) as the sum of an Eisenstein series \( E(\cdot) \) and a cusp form \( f(\cdot) \). Let \( E_m \) and \( f_m \) be the \( m \)th coefficients of the Fourier expansions of \( E(\cdot) \) and \( f(\cdot) \) in the vicinity of \( \infty \). Then
\[ r(m) = E_m + f_m. \]

Here \( E_m \) has the form \( E_m = \sigma(m)\sqrt{m}L(1)/\sqrt{p} \), where \( \sigma(m) \) depends only on the residue of \( m \) modulo \( 24p \) and \( L(s) \) is the Dirichlet \( L \)-series with quadratic character \( (-6/p) \). Under the assumptions of the lemma we have \( \sigma(m) > 0 \). Thus, \( E_m > C_\varepsilon m^{1/2-\varepsilon}/p^{1/2+\varepsilon} \), where \( \varepsilon > 0 \) is arbitrarily small and \( C_\varepsilon > 0 \) is a constant depending on \( \varepsilon \).

If \( m \) is not a square and \( (m, 6p) = 1 \), then it follows from a result of [5] that \( f_m = O(m^{1/2-1/28}(f, f)^{1/2}) \), where \( (\cdot, \cdot) \) is the Petersson scalar product. Proceeding as in [6], we get the estimate \( (f, f) = O(p^{1/2+\varepsilon}) \), where \( \varepsilon > 0 \) is arbitrarily small. Consequently, in this case we have \( f_m = O(m^{1/2-1/28+\varepsilon}p^{1/4+\varepsilon}) \). Thus, for \( m > p^{21.5} \) and sufficiently large \( p \) we have \( E_m \gg f_m \) and \( r(m) > 0 \).

On the other hand, if \( m \) is a square, then Eq. (5) has at least one (trivial) solution.

Proof of the theorem. We consider only those solutions of Eq. (4) for which \( z + u \) is even. Let \( z + u = 2A \). Then Eq. (4) takes the form
\[ n = x^2 + y^2 + 6Az^2 - 12A^2z + 8A^3 + v^4 + w^{14}. \]

Letting \( z_1 = z - A \), we can rewrite this in the form
\[ n = x^2 + y^2 + 6Az_1^2 + 2A^3 + v^4 + w^{14}. \] (6)

Since \( n \equiv 1 \pmod{5} \), the congruence \( n \equiv w_0^4 \pmod{5^{4\alpha}} \) is solvable for every \( \alpha \). Take \( \alpha \) so that \( 5^{4\alpha} \approx c_1 n_1^{1/14} \) with sufficiently small constant \( c_1 \). Set \( w = w_0 + 5^{4\alpha}t_1 \), where \( 0 < w_0 < 5^{4\alpha} \). Then \( n - w^{14} = 5^{4\alpha}n_1 \). The parameter \( t_1 \) can be chosen so that \( n_1 \equiv 1 \pmod{5} \). Note that \( n_1 \approx n^{13/14} \). Set \( x = 5^{2\alpha}x_1, y = 5^{2\alpha}y_1, A = 5^{4\alpha}B, \) and \( v = 5^{\alpha}v_1 \). Then Eq. (6) takes the form
\[ n_1 = x^2 + y^2 + 6Bx_1^2 + 2 \cdot 5^{8\alpha}B^3 + v_1^4. \] (7)

Now we repeat the above reasoning: choose the exponent \( \beta \) so that \( 5^{2\beta} \approx c_2 n_1^{1/4} \), where \( c_2 \) is a sufficiently small constant. Set \( v_1 = v_0 + 5^{2\beta}t_2 \), where \( 0 < v_0 < 5^{2\beta} \) is the smallest solution of the congruence \( n_1 \equiv v_0^4 \pmod{5^{2\beta}} \). Then \( n_1 - v_1^4 = 5^{2\beta}n_2 \). Set \( x_1 = 5^{\beta}x_2, y_1 = 5^{\beta}y_2, \) and \( B = 5^{2\beta}p \). Then Eq. (7) takes the form
\[ n_2 = x_2^2 + y_2^2 + 6pz_1^2 + 2 \cdot 5^{8\alpha+4\beta}p^3. \] (8)