THE REGION OF VALUES OF A SYSTEM OF FUNCTIONALS IN THE CLASS OF TYPICALLY REAL FUNCTIONS

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Let $T_R$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} c_n z^n$ that are regular and typically real in the disk $E = \{z : |z| < 1\}$. For this class, the region of values of the system $\{f(z_0), f(r)\}$ for $z_0 \in \mathbb{R}$, $r \in (-1, 1)$ is studied. The sets $D_r = \{f(z_0) : f \in T_R, f(r) = a\}$ for $-1 \leq r \leq 1$ and $\Delta_r = \{(c_2, c_3) : f \in T_R, -f(-r) = a\}$ for $0 < r \leq 1$ are found, where $a \in (r(1 + r)^{-2}, r(1 - r)^{-2})$ is an arbitrary fixed number. Bibliography: 11 titles.

§1

One of the basic classes considered in the geometric theory of functions is the class $T_R$ of typically real functions, i.e., the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

that are regular in the disk $E = \{z : |z| < 1\}$ and satisfy the following condition: $\text{Im} f(z) \cdot \text{Im} z > 0$ if $\text{Im} z \neq 0$ and $\text{Im} f(z) = 0$ if $\text{Im} z = 0$. The history of study of this class goes back to V. Rogosinski [1], who found the region of values of the system $\{c_2, c_3, \ldots, c_n\}$, $n \geq 2$, as well as of the quantity $f(z_0)$ in the class $T_R$ (see Figs. 1 and 2). The interest in the class $T_R$ is explained, in particular, by the well-known fact that the class $S_R$ of functions of the form (1) with real coefficients $c_n$, $n \geq 2$, that are regular and univalent in $E$ is contained in the class $T_R$, and the closed convex hull of $S_R$ coincides with the class $T_R$.

The integral representation of the class $T_R$ obtained by M. Robertson [2] and G. M. Goluzin [3] has been widely used for solution of extremal problems in this class. For instance, Yu. E. Alenitsyn [4] determined the region of values of the system $\{f(z_0), c_2\}$ in the class $T_R$ and, as a corollary, found the region of values $f(z_0)$ in the subclass of functions from $T_R$ with fixed $c_2$.

To study the regions of values of systems of functionals in the classes of functions representable by the Stieltjes integral some authors used Carathéodory-Toeplitz’ and F. Riesz’ well-known theorems for Carathéodory’s $C$-functions and their generalizations; see N. Akhiezer and M. Krein [5]. The mentioned results were used in [6] for the study of the region of values of a system of the form

$$\{f^{(n)}(z_k)\} \quad (k = 1, 2, \ldots, m; \quad n = 0, 1, 2, \ldots, n_k),$$

where $z_k$ are arbitrary fixed points of $E$, in the classes of functions connected with the $C$-functions. E. G. Goluzina [7] studied the set $\{(f(z_0), c_2, c_3, \ldots, c_n) : f \in T_R\}$, $n \geq 2$ and the region of values of system (2) in the class $T_R$.

When studying the class $T_R(M)$ of bounded typically real functions, V. V. Chernikov [8] established the connection between this class and the class $T_R^{-}(M)$ of functions from $T_R$ satisfying the condition

$$\lim_{r \to -1^+} f(r) \geq -\frac{M}{4}, \quad 1 < M < \infty.$$ 

The Carathéodory theorem for the $C$-functions was used in [8] for finding the set $\{(c_2, c_3) : f \in T_R^{-}(M)\}$ and describing the boundary functions of the region of values of the system $\{c_2, c_3, \ldots, c_n\}$, $n \geq 2$, in the class $T_R^{-}(M)$.

In the present paper we study the set

$$D = D(z_0, r) \equiv \{(f(z_0), f(r)) : f \in T_R\} \quad (z_0 \in E, -1 < r < 1)$$

and complement the result of [8] on the region of values of $\{c_2, c_3\}$ in the class $T_R^{-}(M)$.

Let
\[ r, r_0 \neq 0, \quad -1 < r < 1, \quad -1 < r_0 < 1; \quad 0 < |z_0| < 1; \quad a \in \left( \frac{r}{(1+r)^2}, \frac{r}{(1-r)^2} \right). \]

Set
\[ D = \{(w_1, x_2) \in \mathbb{R}^3 : w_1 = x_1 + iy_1 = f(z_0), \ x_2 = f(r), \ f \in T_R \}, \]
\[ D_r = \{w_1 : w_1 = f(z_0), \ f \in T_R, \ f(r) = a\}, \]
\[ \zeta = z_0 + \frac{1}{z_0}, \quad \rho = r + \frac{1}{r}, \quad \rho_0 = r_0 + \frac{1}{r_0}, \]
\[ \epsilon_1 = 1, \quad \epsilon_2 = -1, \quad \sigma = \text{sign} r. \]

Below we prove the following theorem.

**Theorem 1.** (i) In the case \( \text{Im} z_0 \neq 0 \) the set \( D \) is defined by the system of inequalities
\[
|\epsilon_j + (2 - \rho \epsilon_j)x_2| \left\{ (2 - \rho \epsilon_j)x_2 + 2 \text{Re} \left[ \frac{(2 - \zeta \epsilon_j)(\zeta - \rho)}{\zeta - \overline{\zeta}} w_1 \right] \right\} - |(2 - \rho \epsilon_j)x_2 - (2 - \zeta \epsilon_j)w_1|^2 \geq 0, \quad \sigma |\epsilon_j + (2 - \rho \epsilon_j)x_2| \geq 0, \quad j = 1, 2. \tag{3}
\]

In the case \( \text{Im} z_0 = 0, \ z_0 = r_0 \), \( D \) is bounded by the arc
\[ l = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = (\rho - \rho_0 + x_1^{-1})^{-1}, \ \ (\rho_0 + 2)^{-1} \leq x_1 \leq (\rho_0 - 2)^{-1}\} \]
and the segment \( L \) connecting the endpoints of the arc.

The points of the arc \( l \) correspond to the functions
\[ f(z) = \frac{z}{(1 - 2tz + z^2)}, \quad -1 \leq t \leq 1, \tag{4} \]
and the points of the segment \( L \) correspond to the functions
\[ f(z) = \frac{\lambda z}{(1 - z)^2} + \frac{(1 - \lambda)z}{(1 + z)^2}, \quad 0 \leq \lambda \leq 1. \tag{5} \]

(ii) In the case \( \text{Im} z_0 \neq 0 \) we have \( D_r = C_r^{(1)} \cap C_r^{(2)} \), where
\[ C_r^{(j)} = \left\{ w_1 : \left| w_1 - \frac{(2 - \rho \epsilon_j)(\rho - \zeta)a + \epsilon_j(\rho - \zeta)}{(\zeta - \overline{\zeta})(2 - \zeta \epsilon_j)} \right| \leq \left| (\rho - \zeta)(1 - (\rho - 2\epsilon_j)a) \right| \left| (\zeta - \overline{\zeta})(2 - \zeta \epsilon_j) \right| \right\}, \quad j = 1, 2. \tag{6} \]

The points of the sets \( \partial D_r \cap C_r^{(j)}, \ j = 1, 2, \) correspond to the functions
\[ f_j(z, t_j) = \frac{z(1 + z^2 + [-\rho + (\rho - 2\epsilon_j)(\rho - 2t_j)a]z)}{(1 + z^2 - 2\epsilon_j z)(1 + z^2 - 2t_j z)}, \tag{7} \]
\[ -1 \leq t_1 \leq \frac{p_a - 1}{2a}, \quad \frac{p_a - 1}{2a} \leq t_2 \leq 1. \]

Let \( \delta = \pm 1 \) and let \( D_\delta = \{ f(z_0) : f \in T_R, f(\delta) = a\} \), where \( a \in (1/4, \infty) \) for \( \delta = 1 \) and \( a \in (-\infty, -1/4) \) for \( \delta = -1 \).