CONVOLUTION PROPERTIES OF SOME CLASSES OF ANALYTIC FUNCTIONS

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Let \( A \) denote the class of functions which are analytic in \( |z| < 1 \) and normalized so that \( f(0) = 0 \) and \( f'(0) = 1 \), and let \( \mathcal{R}(\alpha, \beta) \subset A \) be the class of functions \( f \) such that \( \text{Re}[f'(z) + \alpha zf''(z)] > \beta \), \( \text{Re} \alpha > 0 \), \( \beta < 1 \). We determine conditions under which

(i) \( f \in \mathcal{R}(\alpha_1, \beta_1), g \in \mathcal{R}(\alpha_2, \beta_2) \) implies that the convolution \( f \ast g \) of \( f \) and \( g \) is convex;

(ii) \( f \in \mathcal{R}(0, \beta_1), g \in \mathcal{R}(0, \beta_2) \) implies that \( f \ast g \) is starlike;

(iii) \( f \in A \) such that \( f'(z)[f(z)/z]^{\mu-1} < 1 + \lambda z, \mu > 0, 0 < \lambda < 1, \) is starlike, and

(iv) \( f \in A \) such that \( f'(z) + \alpha zf''(z) < 1 + \delta z, \alpha > 0, \delta > 0, \) is convex or starlike.

Bibliography: 16 titles.

1. INTRODUCTION

Let \( U = \{z : |z| < 1\} \) be the unit disk and let \( \mathcal{H} \) be the class of functions analytic in \( U \). Further, let \( \mathcal{A} \subset \mathcal{H} \) be the class of normalized analytic functions \( f \) in \( U \) such that \( f(0) = f'(0) - 1 = 0 \). Let \( S \subset \mathcal{A} \) be the class of normalized univalent functions in \( U \). We need the following classes of functions:

\[
\begin{align*}
B &= \{f \in \mathcal{H} : f(0) = 0, |f(z)| < 1, z \in U \}, \\
\mathcal{P}_\alpha &= \{f \in \mathcal{H} : f(0) = 1, \text{Re} f(z) \geq \alpha, \alpha \leq 1, z \in U \}, \\
S^*(\beta) &= \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \in \mathcal{P}_\beta, \beta \leq 1, z \in U \}, \\
\mathcal{K}(\beta) &= \{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_\beta, \beta \leq 1, z \in U \}, \\
\mathcal{R}_1(\lambda) &= \{f \in \mathcal{A} : |f'(z) - 1| < \lambda, 0 < \lambda \leq 1, z \in U \}, \\
\mathcal{R}(\alpha, \beta) &= \{f \in \mathcal{A} : \text{Re}[f'(z) + \alpha zf''(z)] > \beta, \text{Re} \alpha > 0, \beta < 1, z \in U \}.
\end{align*}
\]

We denote the classes \( \mathcal{P}_0, S^*(0), \mathcal{K}(0), \) and \( \mathcal{R}(0, \beta) \) by \( \mathcal{P}, S^*, \mathcal{K}, \) and \( \mathcal{R}(\beta) \), respectively. The convolution (Hadamard product) of two functions \( f, g \in \mathcal{H} \) of the form

\[
f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n, \quad g(z) = b_0 + \sum_{n=1}^{\infty} b_n z^n
\]

is defined as

\[
h(z) = (f \ast g)(z) = a_0 b_0 + \sum_{n=1}^{\infty} a_n b_n z^n.
\]

Further, an analytic function \( f \in \mathcal{H} \) is said to be subordinate to a univalent function \( g \in \mathcal{H} \) if \( f(z) = g(w(z)) \), \( w \in B \), \( f(0) = g(0) \); this subordination is symbolically denoted by \( f \prec g \) or \( f(z) \prec g(z) \).

The class \( \mathcal{R}(1, 0) \) was studied in [1] and it was shown in [2] that \( \mathcal{R}(1, 0) \subset S^* \).

Theorem A1 [3]. (i) \( \mathcal{R}(1, -1/4) \subset S^* \).

(ii) If \( f \in \mathcal{R}(1, \beta) \), and \( g \in \mathcal{A} \), then \( \text{Re} \left( \frac{g(z)}{z} \right) > 1/2 \) implies \( f \ast g \in \mathcal{R}(1, \beta) \).

(iii) If \( f, g \in \mathcal{R}(1, 0) \) and \( h(z) = (f \ast g)(z) \), then \( zh'(z) \in \mathcal{R}(1, -1/4) \) and hence \( h \in \mathcal{K} \).

We define

\[
\gamma(\alpha) = \inf \{\beta : \mathcal{R}(\alpha, \beta) \subset S^* \}.
\]

The class \( \mathcal{R}(\alpha, \beta) \) has recently been studied by several authors and the following is the best known result.

Theorem A2 [4]. (i) \( R(1, \beta) \subset S^* \) if \( \beta \geq \beta_0 = (1 - 2\log 2)/(2 - 2\log 2) = -0.629 \ldots = \gamma(1) \).
(ii) \( R(1/2, \beta) \subset S^* \) if \( \beta \geq \beta_1 = (3 - 4\log 2)/(2 - 4\log 2) = -0.294 \ldots \).

The given values of \( \beta_0 \) and \( \beta_1 \) are the best possible.

It is known that \( K \ast K \subset K \) (see [5]) and that \( R(0) \not\subset S^* \) (see [6]). In fact, the radius of starlikeness of \( R(0) \) is greater than 0.896 (see [7]). Further, whereas \( R(1, \gamma) \subset S^* \) for some \( \gamma < 0 \) it is known that \( R(1, 0) \not\subset K \) (see [6]). Since \((f \ast g)(z), f, g \in S^* \) need not even be univalent, it is of interest to generalize the convolution properties in Theorem A1 not only to \( R(\alpha, \beta) \), but also to other classes of functions.

Another class which is of interest to us is \( R(\lambda) \).

Theorem A3 [8]. \( R_1(\sqrt{4/5}) \subset S^* \).

It has been shown in [9] that \( \sqrt{4/5} \) is the best possible bound here. One of the results established in the present paper generalizes Theorem A3 in the sense that it gives the order of starlikeness of starlike functions in this class. The proof depends on Lemma 1 (stated below) which is of function theoretic interest and some of its applications are given in the results that follow the lemma.

It may be mentioned that Lemma 1 is a combination of Lemma 3 of [10] and Lemma of [11] which were established by the method of differential subordination. Here a direct function-theoretic proof is given which also sharpens the results of [10].

In our proofs of the results of this paper we heavily rely on the following theorem.

Theorem A4 [12]. If \( f, g \in H \) and \( F, G \in K \) are such that \( f \prec F, g \prec G \), then \( f \ast g \prec F \ast G \).

Theorem A5 [13]. For \( \alpha \leq 1, \beta \leq 1 \)

\[
P_{\alpha} \ast P_{\beta} \subset P_{\delta}, \quad \delta = 1 - 2(1 - \alpha)(1 - \beta).
\]

The result is sharp.

In fact the last theorem is a corollary of Theorem A4. Using these theorems not only yields more precise information but also allows us to avoid using cumbersome coefficient inequalities which often can be tricky. For the problems discussed here, using Theorems A4 and A5 is both natural and elegant.

To establish the sharpness of some of our results we need the following theorem.

Theorem A6 [9]. Let \( \alpha, \beta \in \mathbb{R} \). Then there exists a sequence of functions \( \{f_k\} \) analytic in the closed unit disk and such that \(|f_k(z)| \leq |z|, f_k(1) = e^{i\alpha}, \) and \( \lim_{k \to \infty} f_k(z) = ze^{i\beta} \) uniformly on compact subsets of \( U \).

2. Statements of Results

Theorem 1. For \( \alpha_2 \geq \alpha_1 > 0 \) let \( R(\alpha_1, \beta_1), R(\alpha_2, \beta_2) \subset S^* \) and let \( \beta = \max(\beta_1, \beta_2) \). If \( f \in R(\alpha_1, \beta), g \in R(\alpha_2, \beta), \) and \( h(z) = (f \ast g)(z), \) then \( h \in K \) provided that

\[
1 - \beta_2 \geq 4(1 - \beta)^2(1 - \delta(\alpha_1)),
\]

where

\[
\delta(\alpha_1) = \int_0^1 \frac{dt}{1 + t^\alpha_1}.
\]

It is easy to see from Theorem A2 and the proof of Theorem 1 that the following result holds.

Theorem 1'. (a) \( R(1, \beta) \ast R(1, \beta) \subset K \) for \( \beta \geq -0.152 \ldots \) and
(b) \( R(1/2, \beta) \subset K \) for \( \beta \geq -0.046 \ldots \).

This theorem is an extension and improved version of Theorem A1(iii).